III.

THE INTRODUCTION OF THE NOTION OF FOUR-DIMENSIONAL POINT-AGGREGATES, PERMISSIBLE.

In the preceding section it was shown that we can conceive not only of manifoldnesses of one, two, and three dimensions, but also of manifoldnesses of *any* number of dimensions. But it was at the same time indicated that our world-space, that is, the totality of all conceivable *points* that differ only in respect of position, cannot in agreement with our notions of things possess more than three dimensions. But the question now arises, whether, if the progress of science tends in such a direction, it is permissible to extend the notion of space by the introduction of point-aggregates of more than three dimensions, and to engage in the study of the properties of such creations, although we know that notwithstanding the fact that we may conceptually establish and explore such aggregates of points, yet we cannot picture to ourselves these creations as we do the spatial magnitudes which surround us, that is, the regular three-dimensional aggregates of points.

To show the reader clearly that this question must be answered in the affirmative, that the extension of our notion of space is permissible, although it leads to things which we cannot perceive by our senses, I may call the reader's attention to the fact that in arithmetic we are accustomed from our youth upwards to extensions of ideas, which, accurately viewed, as little admit of graphic conception as a four-dimensional space, that is, a point-aggregate of four dimensions. By his senses man first reaches only the idea of whole numbers - the results of counting. The observation of primitive peoples* and of children clearly proves that the essential decisive factors of counting are these three: First, we abstract, in the counting of things, completely from the individual and characteristic attributes of these things, that is, we consider them as homogeneous. Second, we associate individually with the things which we count

See the essay *Notion and Definition of Number* in this collection.

other homogeneous things. These other things are even now, among uncivilised peoples, the ten fingers of the two hands. They may, however, be simple strokes, or, as in the case of dice and dominoes, black points on a white background. Third, we substitute for the result of this association some concise symbol or word; for example, the Romans substituted for three things counted, three strokes placed side by side, namely: III; but for greater numbers of things they employed abbreviated signs. The Aztecs, the original inhabitants of Mexico, had time enough, it seems, to express all the numbers up to nineteen by equal circles placed side by side. They had abbreviated signs only for the numbers 20, 400, 8000, and so forth. In speaking, some one same sound might be associated with the things counted; but this method of counting is nowadays employed only by clocks: the languages of men since prehistoric times have fashioned concise words for the results of the association in question. From the notion of number, thus fixed as the result of counting, man reached the notion of the addition of two numbers, and thence the notion that is the inverse of the last process, the notion of subtraction. But at this point it clearly appears that not every problem which may be propounded is soluble; for there is no number which can express the result of the subtraction of a number from one which is equally large or from one which is smaller than itself. The primary school pupil who says that 8 from 5 "won't go" is perfectly right from his point of view. For there really does not exist any result of counting which added to eight will give five.

If humanity had abided by this point of view and had rested content with the opinion that the problem "5 minus 8" is not solvable, the science of arithmetic would never have received its full development, and humanity would not have advanced as far in civilisation as it has. Fortunately, men said to themselves at this crisis: "If 5 minus 8 won't go, we'll *make it go*; if 5 minus 8 does not possess an intelligible meaning, we will simply give it one." As a fact, things which have not a meaning always afford men a pleasing opportunity of investing them with one. The question is, then, what significance is the problem "5 minus 8" to be invested with?

The most natural and therefore the most advantageous solution undoubtedly is to abide by the original notion of subtraction as the inverse of addition, and to make the significance of 5 minus 8 such, that for 5 minus 8 plus 8 we shall get our original minuend 5. By such a method all the rules of computation which apply to real differences will also hold good for unreal differences, such as 5 minus 8. But it then clearly appears that all forms expressive of differences in which the numbers that stand before the minus symbol are less by an equal amount than those which follow it may be regarded as equal; so that the simplest course seems to be to introduce as the common characteristic of all equal differential forms of this description a common sign, which will indicate at the same time the difference of the two numbers thus associated. Thus it came about that for 5 minus 8, as well as for every differential form which can be regarded as equal thereto, the sign "-3" was introduced. But in calling differential forms of this description numbers, the notion of number was extended and a new domain was opened up, namely the domain of negative numbers.

In the further development of the science of arithmetic, through the operation of division viewed as the inverse of multiplication, a second extension of the idea of number was reached, namely, the notion of fractional numbers as the outcome of divisions that had led to numbers hitherto undefined. We find, thus, that the science of arithmetic throughout its whole development has strictly adhered to the principle of conformity and consistency and has invested every association of two numbers, which before had no significance, by the introduction of new numbers, with a real significance, such that similar operations in conformity with exactly the same rules could be performed with the new numbers, viewed as the results of this association, as with the numbers which were before known and perfectly defined. Thus the science proceeded further on its way and reached the notions of irrational, imaginary, and complex numbers.

The point in all this, which the reader must carefully note, is, that all the numbers of arithmetic, with the exception of the positive whole numbers, are artificial products of human thought, invented to make the language of arithmetic more flexible, and to

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accelerate the progress of science. All these numbers lack the attributes of representability.

No man in the world can picture to himself "minus three trees." It is possible, of course, to know that when three trees of a garden have been cut down and carried away, three are missing, and by substituting for "missing" the inverse notion of "added," we may say, perhaps, that "minus three trees" are added. But this is quite different from the feat of imagining a negative number of trees. We can only picture to ourselves a number of trees that results from actual counting, that is, a positive whole number. Yet, notwithstanding all this, people had not the slightest hesitation in extending the notion of number. Exactly so must it be permitted us in geometry to extend the notion of space, even though such an extension can only be mentally defined and can never be brought within the range of human powers of representation.

In mathematics, in fact, the extension of any notion is admissible, provided such extension does not lead to contradictions with itself or with results which are well established. Whether such extensions are necessary, justifiable, or important for the advancement of science is a different question. It must be admitted, therefore, that the mathematician is justified in the extension of the notion of space as a point-aggregate of three dimensions, and in the introduction of space or point-aggregates of more than three dimensions, and in the employment of them as means of research. Other sciences also operate with things which they do not know exist, and which, though they are sufficiently defined, cannot be perceived by our senses. For example, the physicist employs the ether as a means of investigation, though he can have no sensory knowledge of it. The ether is nothing more than a means which enables us to comprehend mechanically the effects known as action at a distance and to bring them within the range of a common point of view. Without the assumption of a material which penetrates everything, and by means of whose undulations impulses are transmitted to the remotest parts of space, the phenomena of light, of heat, of gravitation, and of electricity would be a jumble of isolated and unconnected mysteries. The assumption of an ether, however, comprises

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in a systematic scheme all these isolated events, facilitates our mental control of the phenomena of nature, and enables us to produce these phenomena at will. But it must not be forgotten in such reflexions that the ether itself is even a greater problem for man, and that the ether-hypothesis does not solve the difficulties of phenomena, but only puts them in a unitary conceptual shape. Notwithstanding all this, physicists have never had the least hesitation in employing the ether as a means of investigation. And as little do reasons exist why the mathematicians should hesitate to investigate the properties of a four-dimensioned point-aggregate, with the view of acquiring thus a convenient means of research.

IV.

THE INTRODUCTION OF THE IDEA OF FOUR-DIMENSIONED POINT-AGGREGATES OF SERVICE TO RESEARCH.

From the concession that the mathematician has the right to define and investigate the properties of point-aggregates of more than three dimensions, it does not necessarily follow that the introduction of an idea of this description is of value to science. Thus, for example, in arithmetic, the introduction of operations which spring from involution, as involution and its two inverse operations proceed from multiplication, is undoubtedly permitted. Just as for "a times a times a" we write the abbreviated symbol "a^3," (which we read, a to the third power,) and investigate in detail the operation of involution thus defined, so we might also introduce some shorthand symbol for "a to the ath power to the ath power" and thus reach an operation of the fourth degree, which would regard a as a passive number and the number 3, or any higher number, as the active number, that is, as the number which indicates how often a is taken as the base of a power whose exponent may be a, or "a to the ath" or "a to the a" to the a" power."

But the introduction of such an operation of the fourth degree has proved itself to be of no especial value to mathematics. And the reason is that in the operation of involution the law of commutation does not hold good. In addition, the numbers to be added may be interchanged and the introduction of multiplication is therefore

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of great value. So, also, in multiplication the numbers which are combined, that is the factors, may be changed about in any way, and thus the introduction of involution is of value. But in involution the base and the exponent cannot be interchanged, and consequently the introduction of any higher operation is almost valueless.

But with the introduction of the idea of point-aggregates of multiple dimensions the case is wholly different. The innovation in question has proved itself to be not only of great importance to research, but the progress of science has irresistibly forced investigators to the introduction of this idea, as we shall now set forth in detail.

In the first place, algebra, especially the algebraical theory of systems of equations, derives much advantage from the notion of multi-dimensioned spaces. If we have only three unknown quantities, x, y, z, the algebraical questions which arise from the possible problems of this class admit, as we have above seen, of geometrical representation to the eye. Owing to this possibility of geometrical representation, some certain simple geometrical ideas like "moving," "lying in," "intersecting," and so forth, may be translated into algebraical events. Now, no reason exists why algebra should stop at three variable quantities; it must in fact take into consideration any number of variable quantities.

For purposes of brevity and greater evidentness, therefore, it is quite natural to employ geometrical forms of speech in the consideration of more than three variables. But when we do this, we assume, perhaps without really intending to do so, the idea of a space of more than three dimensions. If we have four variable quantities, x, y, z, u, we arrive, by conceiving attributed to each of these four quantities every possible numerical magnitude, at a four-dimensioned manifoldness of numerical quantities, which we may just as well regard as a four-dimensioned aggregate of points. Two equations which exist on this supposition between x, y, *z,* and u, define two three-dimensioned aggregates of points, which intersect, as we may briefly say, in a two-dimensioned aggregate of points, that is, in a surface; and so on. In a somewhat different manner the determination of the contents of a square or a cube by the involution

of a number which stands for the length of its sides, leads to the notion of four-dimensioned structures, and, consequently, to the notion of a four-dimensioned point-space. When we note that a^2 stands for the contents of a square, and a^3 for the content of a cube, we naturally inquire after the contents of a structure which is produced from the cube as the cube is produced from the square and which also will have the contents a^4. We cannot, it is true, clearly picture to ourselves a structure of this description, but we can, nevertheless, establish its properties with mathematical exactness.* It is bounded by 8 cubes just as the cube is bounded by 6 squares; it has 16 corners, 24 squares, and 32 edges, so that from every corner 4 edges, 6 squares, and 4 cubes proceed, and from every edge 3 squares and 3 cubes.

Yet despite the great service to algebra of this idea of multi-dimensioned space, it must be conceded that the conception although convenient is yet not indispensable. It is true, algebra is in need of the idea of multiple dimensions, but it is not so absolutely in need of the idea of *point*-aggregates of multiple dimensions.

This notion is, however, necessary and serviceable for a profound comprehension of geometry. The system of geometrical knowledge which Euclid of Alexandria created about three hundred years before Christ, supplied during a period of more than two thousand years a brilliant example of a body of conclusions and truths which were mutually consistent and logical. Up to the present century the idea of elementary geometry was indissolubly bound up with the name of Euclid, so that in England where people adhered longest to the rigid deductive system of the Grecian mathematician, the task of "learning geometry" and "reading Euclid" were until a few years ago identical. Every proposition of this Euclidean system rests on other propositions, as one building-stone in a house rests upon another. Only the very lowest stones, the foundations, were without supports. These are the axioms or fun-

*Victor Schlegel, indeed, has made models of the three-dimensional nets of all the six structures which correspond in four-dimensioned space to the five regular bodies of our space, in an analogous manner to that by which we draw in a plane the net of five regular bodies. Schlegel's models are made by Brill of Darmstadt.

damental propositions, truths on which all other truths are, directly or indirectly, founded, but which themselves are assumed without demonstration as self-evident.

But the spirit of mathematical research grew in time more and more critical, and finally asked, whether these axioms might not possibly admit of demonstration. Especially was a rigid proof sought for the eleventh * axiom of Euclid, which treats of parallels.

After centuries of fruitless attempts to prove Euclid's eleventh axiom, Gauss, and with him Bolyai and Lobachévski, Riemann, and Helmholtz, finally stated the decisive reasons why any attempt to prove the axiom of the parallels must necessarily be futile. These reasons consist of the fact that though this axiom holds good enough in the world-space such as we do and can conceive it, yet three-dimensioned spaces are ideally conceivable though not capable of mental representation, where the axiom does not hold good. The axiom was thus shown to be a mere fact of *observation,* and from that time on there could no longer be any thought of a deductive demonstration of it. In view of the intimate connection, which both in an historical and epistemological point of view exists between the extension of the concept of space and the critical examination of the axioms of Euclid, we must enter at somewhat greater length into the discussion of the last mentioned propositions.

Of the axioms which Euclid lays at the foundation of his geometry, only the following three are really geometrical axioms:

Eighth axiom: Magnitudes which coincide with one another are equal to one another.

Eleventh axiom: If a straight line meet two straight lines so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Twelfth axiom: Two straight lines cannot inclose a [finite] space.

The numerous proofs which in the course of time were adduced

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*Also called the twelfth axiom, also the fifth postulate. - *Tr.*

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in demonstration of these axioms, especially of the eleventh, all turn out on close examination to be pseudo-proofs. Legendre drew attention to the fact that either of the following axioms might be substituted for the eleventh:

a) Given a straight line, there can be drawn through a point in the same plane with that line, one and one line only which shall not intersect the first (parallels) however far the two lines may be produced;

b) If two parallel lines are cut by a third straight line, the interior alternate angles will be equal.

c) The sum of the angles of a triangle is equal to two right angles, that is, to the angle of a straight line or to 180^o.

By the aid of any one of these three assertions, the eleventh axiom of Euclid may be proved, and, *vice versa,* by the aid of the latter each of the three assertions may be proved, of course with the help of the other two axioms, eight and twelve. The perception that the eleventh axiom does not admit of demonstration without the employment of one of the foregoing substitutes may best be gained from the consideration of congruent figures. Every reader will remember from his first instruction in geometry that the congruence of two triangles is demonstrated by the superposition of one triangle on the other and by then ascertaining whether the two completely coincide, no assumptions being made in the determination except those above mentioned.

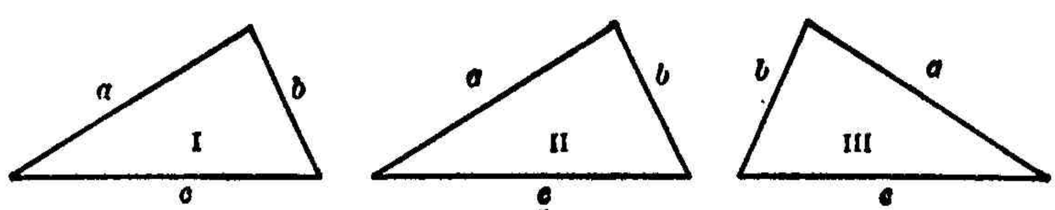

Fig. 35.

In the case of triangles which are congruent, as are I and II in the preceding cut, this coincidence may be effected by the simple *displacement* of one of the triangles; so that even a two-dimensional being, supposed to be endowed with powers of reasoning, but only capable of picturing to itself motions within a plane, also might convince itself that the two triangles I and II could be made to coincide. But a being of this description could not convince itself

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in like manner of the congruence of triangles I and III. It would discover the equality of the three sides and the three angles, but it could never succeed in so superposing the two triangles on each other as to make them coincide. A three-dimensional being, however, can do this very easily. It has simply to turn triangle I about one of its sides and to shove the triangle, thus brought into the position of its reflexion in a mirror, into the position of triangle III. Similarly, triangles II and III may be made to coincide by moving either out of the plane of the paper around one of its sides as axis and turning it until it again falls in the plane of the paper. The triangle thus turned over can then be brought into the position of the other.

Later on we shall revert to these two kinds of congruence: "congruence by displacement" and "congruence by circumversion." For the present we will start from the fact that it is always possible within the limits of a plane to take a triangle out of one position and bring it into another without altering its sides and angles. The question is, whether this is only possible in the plane, or whether it can also be done on other surfaces.

We find that there are certain surfaces in which this is possible, and certain others in which it is not. For instance, it is impossible to move the triangle drawn on the surface of an egg into some other position on the egg's surface without a distension or contraction of some of the triangle's parts. On the other hand, it is quite possible to move the triangle drawn on the surface of a sphere into any other position on the sphere's surface without a distension or contraction of its parts. The mathematical reason of this fact is, that the surface of a sphere, like the plane, has everywhere the same curvature, but that the surface of an egg at different places has different curvatures. Of a plane we say that it has everywhere the curvature zero; of the surface of a sphere we say it has everywhere a positive curvature, which is greater in proportion as the radius is smaller. There are surfaces also which have a constant negative curvature; these surfaces exhibit at every point in directions proceeding from the same side a partly concave and a partly convex structure, somewhat like the centre of a saddle.

There is no necessity of our entering in any detail into the character and structure of the last-mentioned surfaces.

Intimately related with the plane, however, are all those surfaces, which, like the plane, have the curvature zero; in this category belong especially cylindrical surfaces and conical surfaces. A sheet of paper of the form of the sector of a circle may, for example, be readily bent into the shape of a conical surface. If two congruent triangles, now, be drawn on the sheet of paper, which may by displacement be translated the one into the other, these triangles will, it is plain, also remain congruent on the conical surface; that is, on the conical surface also we may displace the one into the other; for though a bending of the figures will take place, there will be no distension or contraction. Similarly, there are surfaces which, like the sphere, have everywhere a constant positive curvature. On such surfaces also every figure can be transferred into some other position without distension or contraction of its parts. Accordingly, on all surfaces thus related to the plane or sphere, the assumption which underlies the eighth axiom of Euclid, that it is possible to transfer into any new position any figure drawn on such surfaces without distortion, holds good.

The eleventh axiom in its turn also holds good on all surfaces of constant curvature, whether the curvature be zero or positive; only in such instances instead of "straight" line we must say "shortest" line. On the surface of a sphere, namely, two shortest lines, that is, arcs of two great circles, always intersect, no matter whether they are produced in the direction of the side at which the third arc of a great circle makes with them angles less than two right angles, or, in the direction of the other side, where this arc makes with them angles of more than two right angles. On the plane, however, two straight lines intersect only on the side where a third straight line that meets them makes with them interior angles less than two right angles.

The twelfth axiom of Euclid, finally, only holds good on the plane and on the surfaces related to it, but not on the sphere or other surfaces which, like the sphere, have a constant positive curvature. This also accounts for the fact that one of the three postulates

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which we regarded as substitutes for the eleventh axiom, though valid for the plane, is not true for the surface of a sphere; namely, the postulate that defines the sum of the angles of a triangle. This sum in a plane triangle is two right angles; in a spherical triangle it is more than two right angles, the spherical triangle being greater, the greater the excess the sum of its angles is above two right angles. It will be seen, from these considerations, that in geometries in which curved surfaces and not fixed planes are studied, the axioms of Euclid are either all or partially false.

The axioms of geometry thus having been revealed as facts of experience, the question suggested itself whether in the same way in which it was shown that different two-dimensional geometries were possible, also different three-dimensional systems of geometry might not be developed; and consequently what the relations were in which these might stand to the geometry of the space given by our senses and representable to our mind. As a fact, a three-dimensional geometry can be developed, which like the geometry of the surface of an egg will exclude the axiom that a figure or body can be transferred from any one part of space to any other and yet remain congruent to itself. Of a three-dimensional space in which such a geometry can be developed we say, that it has no constant measure of curvature.

The space which is representable to us, and which we shall henceforth call the *space of experience,* possesses, as our experiences without exception confirm, the especial property that every bodily thing can be transferred from any one part of it to any other without suffering in the transference any distension or any contraction. The space of experience, therefore, has a constant measure of curvature. The question, however, whether this measure of curvature is zero or positive, that is, whether the space of experience possesses the properties which in two-dimensional structures a plane possesses, or whether it is the three dimensional analogue of the surface of a sphere is one which future experience alone can answer. If the space of experience has a constant positive measure of curvature which is different from zero, be the difference ever so slight, a point which should move forever onward in a straight line, or, more ac-

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curately expressed, in a shortest line, would sometime, though perhaps after having traversed a distance which to us is inconceivable, ultimately have to arrive from the opposite direction at the place from which it set out, just as a point which moves forever onward in the same direction on the surface of a sphere must ultimately arrive at its starting point, the distance it traverses being longer the greater the radius of the sphere or the smaller its curvature.

It will seem, at first blush, almost incredible, that the space of experience possibly could have this property. But an example, which is the historical analogue of this modern transformation of our conceptions, will render the idea less marvellous. Let us transport ourselves to the age of Homer. At that time people believed that the earth was a great disc surrounded on all sides by oceans which were conceived to be in all directions infinitely great. Indeed, for the primitive man, who has never journeyed far from the place of his birth, this is the most natural conception. But imagine now that some scholar had come, and had informed the Homeric hero Ulysses that if he would travel forever on the earth in the same direction he would ultimately come back to the point from which he started; surely Ulysses would have gazed with as much astonishment upon this scholar as we now look upon the mathematician who tells us that it is possible that a point which moves forever onward in space in the same direction may ultimately arrive at the place from which it started. But despite the fact that Ulysses would have regarded the assertion of the scholar as false because contradictory to his familiar conceptions, that scholar, nevertheless, would have been right; for the earth is not a plane but a spherical surface. So also the mathematician may be right who bases this more recent strange view on the possible fact that the space of experience may have a measure of curvature which is not exactly zero but slightly greater than zero. If this were really the case, the *volume* of the space of experience, though very large, would, nevertheless, be finite; just as the real spherical surface of the earth as contrasted with the Homeric plane surface is finite, having so and so many square miles. When the objection is here made that a finiteness of space is totally at variance with our modes of thought and conceptions, two ideas, "infinitely great" and "unlimited,"" are confounded. All that is at variance with our practical conceptions is that space can anywhere have a boundary; not that it may possibly be of tremendous but finite magnitude.

It will now be asked if we cannot determine by actual observation whether the measure of curvature of experiential space is exactly zero or slightly different therefrom. The theorem of the sum of the angles of a triangle and the conclusions which follow from this theorem do indeed supply us with a means of ascertaining this fact. And the results of observation have been, that *the measure of curvature of space is in all probability exactly equal to zero or if it is slightly different from zero it is so little so that the technical means of observation at our command and especially our telescopes are not competent to determine the amount of the deviation*. More, we cannot with any certainty say.

All these reflections, to which the criticism of the hypotheses that underlie geometry long ago led investigators, compel us to institute a comparison between the space of experience and other three-dimensional aggregates of points (spaces), which we cannot mentally represent but can in thought and word accurately define and investigate. As soon, however, as we are fully implicated in the task of accurately investigating the properties of three-dimensional aggregates of points, we find ourselves similarly forced to regard such aggregates as the component elements of a manifoldness of more than three dimensions. In this way the exact criticism of even ordinary geometry leads us to the abstract assumption of space of more than three dimensions. And as the extension of every idea gives a clearer and more translucent form to the idea as it originally stood, here too the idea of multi-dimensioned aggregates of points and the investigation of their properties has thrown a new light on the truths of ordinary geometry and placed its properties in clearer relief. Among the numerous examples which show how the notion of a space of multiple dimensions has been of great service to science in the investigation of three-dimensional space, we shall give one a place here which is within the comprehension of non-mathematicians.

Imagine in a plane two triangles whose angles arc denoted by pairs of numbers - namely, by 1-2, 1-3, 1-4, and 2-5, 3-5, 4-5. (See Fig. 36.) Let the two triangles so lie that the three lines which join the angles 1-2 and 2-5, 1-3 and 3-5, and 1-4 and 4-5 intersect at a point, which we will call 1-5. If now we cause the sides of the triangles which are opposite to these angles to intersect, it will be found that the points of intersection so obtained possess the peculiar property of lying all in one and the same straight line. The point of intersection of the connection 1-3 and 1-4 with the connection 4-5 and 3-5 may appropriately be called 3-4. Similarly, the point of intersection 2-4 is produced by the meeting of 4-5, 2-5 and 1-2, 1-4; and the point of intersection 2-3, by the meeting of 1-3, 1-2 and 3-5, 2-5. The statement, that the three points of intersection 3-4, 2-4, 2-3, thus obtained, lie in one straight line, can be proved by the principles of plane geometry only with difficulty and great circumstantiality. But by resorting to the three- dimensional space of experience, in which the plane of the drawing lies, the proposition can he rendered almost self-evident.

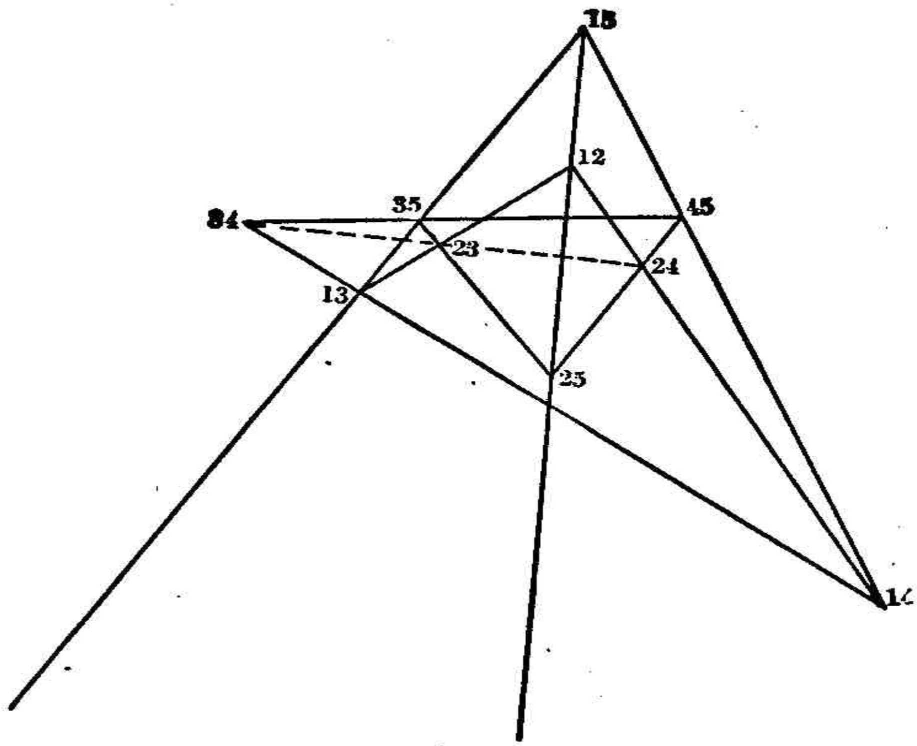

Fig. 36.

To begin with, imagine any five points in space which may be denoted by the numbers I, 2, 3, 4, 5; then imagine all the possible ten straight lines of junction drawn between each two of these points, namely, 1-2, 1-3 . . . . 4-5; and finally, also, all the ten planes of junction of every three points described, namely, the plane 1-2-3, 1-2-4, ... 3-4-5. A spatial figure will thus be obtained, whose ten straight lines will meet some interposed plane in ten points whose relative positions are exactly those of the ten points above described.

Thus, for example, on this plane the points 1-2, 1-3, and 2-3 will lie in a straight line, for through the three spatial points 1, 2, 3, a plane can be drawn which will cut the plane of a drawing in a straight line. The reason, therefore, that the three points 3-4, 2-4, 2-3, also must ultimately lie in a straight line, consists in the simple fact that the plane of the three points 2, 3, 4, must cut the plane of the drawing in a straight line. The figure here considered consists of ten points of which sets of three so lie ten times in a straight line that conversely from every point also three straight lines proceed.

Now, just as this figure is a section of a complete three-dimensional pentagon, so another remarkable figure, of similar proper-

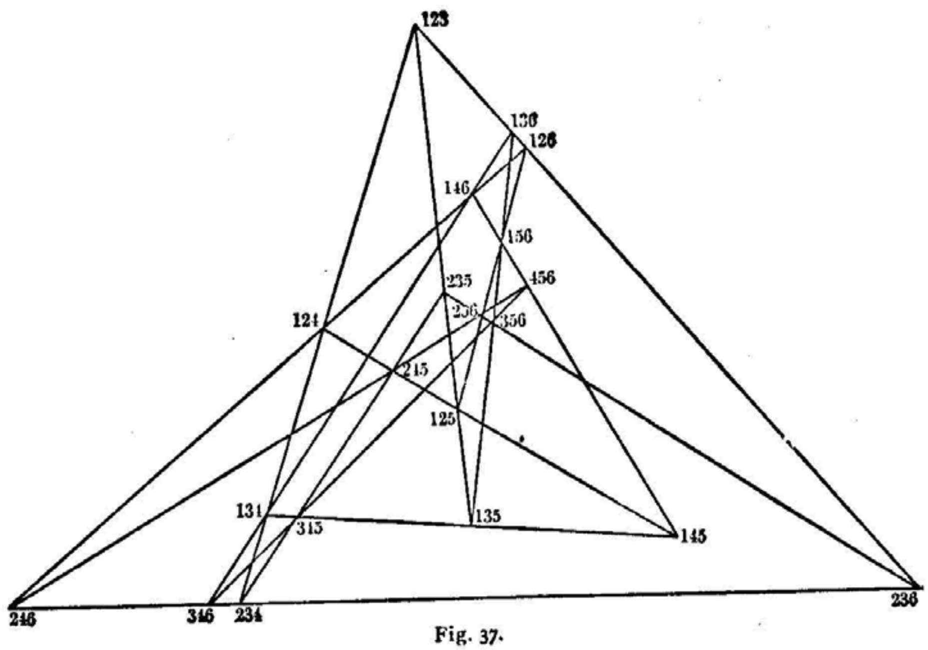

Fig. 37.

ties, may be obtained from the section of a figure of four-dimensioned space. Imagine six points, 1, 2, 3, 4, 5, 6, situated in this four-dimensioned space, and every three of them connected by a plane, and every four of them by a three-dimensioned space. We shall obtain thus twenty planes and fifteen three-dimensioned spaces which will cut the plane in which the figure is to be produced in twenty points and fifteen rays which so lie that each point sends out three rays and every ray contains four points. (See Fig. 37.) Figures of this description, which are so composed of points and rays that an equal number of rays proceed from every point and an equal

number of points lie in every ray, are called *configurations*. Other configurations may, of course, be produced, by taking a different number of points and by assuming that the points taken lie in a space of different or even higher dimensions. The author of this article was the first to draw attention to configurations derived from spaces of higher dimensions. As we see, then, the notion of a space of more than three dimensions has performed an important service in the investigations of common plane geometry.

In conclusion, I should like to add a remark which Cranz makes regarding the application of the idea of multi-dimensioned space to theoretical chemistry. (See the treatise before cited.) In chemistry, the molecules of a compound body are said to consist of the atoms of the elements which are contained in the body, and these are supposed to be situated at certain distances from one another, and to be held in their relative positions by certain forces. At first, the centres of the atoms were conceived to lie in one and the same plane. But Wislicenus was led by researches in paralactic acid to explain the differences of isomeric molecules of the same structural formulation by the different positions of the atoms in *space.* (Compare *La chimie dans l'espace* by van't Hoff, 1875, preface by J. Wislicenus). In fact four points can always be so arranged in space that every two of them may have any distance from each other; and the change of one of the six distances does not necessarily involve the alteration of any other.

But suppose our molecule consists of five atoms? Four of these may be so placed that the distance between any two of them can be made what we please. But it is no longer possible to give the fifth atom a position such that each of the four distances by which it is separated from the other atoms may be what we please. On the contrary, the fourth distance is dependent on the three remaining distances; for the space of experience has only three dimensions. If, therefore, I have a molecule which consists of five atoms I cannot alter the distance between two of them without at least altering some second distance. But if we imagine the centres of the atoms placed in a four-dimensioned space, this can be done; all the ten distances which may be conceived to exist between the five points will then be independent of one another. To reach the same result in the case of six atoms we must assume a five-dimensional space; and so on.

Now, if the independence of all the possible distances between the atoms of a molecule is absolutely required by theoretical chemical research, the science is really compelled, if it deals with molecules of more than four atoms, to make use of the idea of a space of more than three dimensions. This idea is, in this case, simply an instrument of research, just as are, also, the ideas of molecules and atoms - means designed to embrace in a perspicuous and systematic form the phenomena of chemistry and to discover the conditions under which new phenomena can be evoked. Whether a four-dimensioned space really exists is a question whose insolubility cannot prevent research from making use of the idea, exactly as chemistry has not been prevented from making use of the notion of atom, although no one really knows whether the things we call atoms exist or not.

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Schubert - Part 1

Schubert - Part 3

Top of Page

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