Editor's Introduction to

The Fourth Dimension

By Hermann Schubert

Schubert looks at the geometrical foundations of a possible four-dimensional physical space and relates this concept to the modern spiritualism movement of the late nineteenth century. This essay was originally published in English translation in *The Monist* and later republished in Schubert's book *Mathematical Essays and Recreations* of 1899.

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THE FOURTH DIMENSION.

MATHEMATICAL AND SPIRITUALISTIC.

I.

INTRODUCTORY.

THE tendency to generalise long ago led mathematicians to extend the notion of three-dimensional space, which is the space of sensible representation, and to define aggregates of points, or spaces, of more than three dimensions, with the view of employing these definitions as useful means of investigation. They had no idea of requiring people to imagine four-dimensional things and worlds, and they were even still less remote from requiring them to believe in the real existence of a four-dimensioned space. In the hands of mathematicians this extension of the notion of space was a mere means devised for the discovery and expression, by shorter and more convenient ways, of truths applicable to common geometry and to algebra operating with more than three unknown quantities. At this stage, however, the spiritualists came in, and coolly took possession of this private property of the mathematicians. They were in great perplexity as to where they should put the spirits of the dead. To give them a place in the world accessible to our senses was not exactly practicable. They were compelled, therefore, to look around after some *terra incognita,* which should oppose to the spirit of research inborn in humanity an insuperable barrier. The abiding-place of the spirits had perforce to be inaccessible to the senses and full of mystery to the mind. This property the four-dimensioned space of the mathematicians possessed. With an intellectual perversity which science has no idea of, these spiritualists boldly asserted, first, that the whole world was

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situated in a four-dimensioned space as a plane might be situated in the space familiar to us, secondly, that the spirits of the dead lived in such a four-dimensioned space, thirdly, that these spirits could accordingly act upon the world and, consequently, upon the human beings resident in it, exactly as we three-dimensioned creatures can produce effects upon things that are two-dimensioned; for example, such effects as that produced when we shatter a lamina of ice, and so influence some possibly existing two-dimensioned *ice*-world.

Since spiritualism, under the leadership of a Leipsic Professor, Zöllner, thus proclaimed the existence of a four-dimensioned space, this notion, which the mathematicians are thoroughly master of, - for in all their operations with it, though they have forsaken the path of actual representability, they have never left that of the truth, - this notion has also passed into the heads of lay persons who have used it as a catchword, ordinarily without having any clear idea of what they or any one else mean by it. To clear up such ideas and to correct the wrong impressions of cultured people who have not a technical mathematical training, is the purpose of the following pages. A similar elucidation was aimed at in the tracts which Schlegel (Riemann, Berlin, 1888) and Cranz (Virchow-Holtzendorff's Sammlung, Nos. 112 and 113) have published on the so-called fourth dimension. Both treatises possess indubitable merits, but their methods of presentation are in many respects too concise to give to lay minds a profound comprehension of the subject. The author, accordingly, has been able to add to the reflections which these excellent treatises offer, a great deal that appears to him necessary for a thorough explanation in the minds of non-mathematicians of the notion of the fourth dimension.

II.

THE CONCEPT OF DIMENSION.

Many text-books of stereometry begin with the words: "Every body has three dimensions, length, breadth, and thickness." If we should ask the author of a book of this description to tell us the length, breadth, and thickness of an apple, of a sponge, or of a cloud of tobacco smoke, he would be somewhat perplexed and would prob-

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ably say, that the definition in question referred to something different. A cubical box, or some similar structure, whose angles are all right angles and whose bounding surfaces are consequently all rectangles is the only body of which it can at all be unmistakably asserted that there are three principal directions distinguishable in it, of which any one can be called the length, any other the breadth, and any third the thickness. We thus see that the notions of length, breadth, and thickness are not sufficiently clear and universal to enable us to derive from them any idea of what is meant when it is said that every body possesses three dimensions, or that the space of the world is three-dimensional.

This distinction may be made sharper and more evident by the following considerations: We have, let us suppose, a straight line on which a point is situated, and the problem is proposed to determine the position of the point on the line in an unequivocal manner. The simplest way to solve this is, to state how far the point is removed in the one or the other direction from some given fixed point; just as in a thermometer the position of the surface of the mercury is given by a statement of its distance in the direction of cold or heat from a predetermined fixed point - the point of freezing water. To state, therefore, the position of a point on a straight line, the sole datum necessary is a single number, if beforehand we have fixed upon some standard line, like the centimetre, and some definite point to which we give the value zero, and have also previously decided in what direction from the zero-point, points must be situated whose position is expressed by positive numbers, and also in what direction those must lie whose position is expressed by negative numbers. This last-mentioned fact, that a *single* number is sufficient to determine the place of a point in a straight line, is the real reason why we attribute to the straight line or to any part of it a single dimension.

More generally, we call every totality or system, of infinitely numerous things, one-dimensional, in which *one* number is all that is requisite to determine and mark out any particular one of these things from among the entire totality. Thus, time is one-dimensional. We, as inhabitants of the earth, have naturally chosen as

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our unit of time, the period of the rotation of the earth about its axis, namely, the day, or a definite portion of a day. The zero-point of time is regarded in Christian countries as the year of the birth of Christ, and the positive direction of time is the time *subsequent* to the birth of Christ. These data fixed, all that is necessary to establish and distinguish any definite point of time amid the infinite totality of all the points of time, *is a single number*. Of course this number need not be a whole number, but may be made up of the sum of a whole number and a fraction in whose numerator and denominator we may have numbers as great as we please. We may, therefore, also say that the totality of all conceivable numerical magnitudes, or of only such as are greater than one definite number and smaller than some other definite number, is one-dimensional.

We shall add here a few additional examples of one-dimensional magnitudes presented by geometry. First, the circumference of a circle is a one-dimensional magnitude, as is every curved line, whether it returns into itself or not. Further, the totality of all equilateral triangles which stand on the same base is one-dimensional, or the totality of all circles that can be described through two fixed points. Also, the totality of all conceivable cubes will be seen to be one-dimensional, provided they are distinguished, not with respect to position, but with respect to magnitude.

In conformity with the fundamental ideas by which we define the notion of a one-dimensional manifoldness, it will be seen that the attribute two-dimensional must be applied to all totalities of things in which *two* numbers are necessary (and sufficient) to distinguish any determinate individual thing amid the totality. The simplest two-dimensioned complex which we know of is the plane. To determine accurately the position of a point in a plane, the simplest way is to take two axes at right angles to each other, that is, fixed straight lines, and then to specify the distances by which the point in question is removed from each of these axes.

This method of determining the position of a point in a plane suggested to the celebrated philosopher and mathematician Descartes the fundamental idea of analytical geometry, a branch of mathematics in which by the simple artifice of ascribing to every

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point in a plane two numerical values, determined by its distances from the two axes above referred to, planimetrical considerations are transformed into algebraical. So, too, all kinds of curves that graphically represent the dependence of things on time, make use of the fact that the totality of the points in a plane is two-dimensional. For example, to represent in a graphical form the increase in the population of a city, we take a horizontal axis to represent the time, and a perpendicular one to represent the numbers which are the measures of the population. Any two lines, then, whose lengths practical considerations determine, are taken as the unit of time, which we may say is a year, and as the unit of population, which we will say is one thousand. Some definite year, say 1850, is fixed upon as the zero point. Then, from all the equally distant points on the horizontal axis, which points stand for the years, we proceed in directions parallel to the other axis, that is, in the perpendicular direction, just so much upwards as the numbers which stand for the population of that year require. The terminal points so reached, or the curve which runs through these terminal points, will then present a graphic picture of the rates of increase of the population of the town in the different years. The rectangular axes of Descartes are employed in a similar way for the construction of barometer curves, which specify for the different localities of a country the amount of variation of the atmospheric pressure during any period of time. Immediately next to the plane the surface of the earth will be recognised as a two-dimensional aggregate of points. In this case geographical latitude and longitude supply the two numbers that are requisite accurately to determine the position of a point. Also, the totality of all the possible straight lines that can be drawn through any point in space is two-dimensional, as we shall best understand if we picture to ourselves a plane which is cut in a point by each of these straight lines and then remember that by such a construction every point on the plane will belong to some one line and, *vice versa,* a line to every point, whence it follows that the totality of all the straight lines, or, as we may call them, rays, which pass through the point assigned are of the same dimensions as the totality of the points of the imagined plane.

The question might be asked, In what way and to what extent in this case is the specification of *two* numbers requisite and sufficient to determine amid all the rays which pass through the specified point a definite individual ray? To get a clear idea of the problem here involved, let us imagine the ray produced far into the heavens where some quite definite point will correspond to it. Now, the position of a point in the heavens depends, as does the position of a point on all spherical surfaces, on two numbers. In the heavens these two numbers are ordinarily supplied by the two angles called altitude, or the distance above the plane of the horizon, and azimuth, or the angular distance between the circle on which the altitude is measured and the meridian of the observer. It will be seen thus that the totality of all the luminous rays that an eye, conceived as a point, can receive from the outer world is two-dimensional, and also that a luminous point emits a two-dimensional group of luminous rays. It will also be observed, in connexion with this example, that the two-dimensional totality of all the rays that can be drawn through a point in space is something different from the totality of the rays that pass through a point but are required to lie in a given plane. Such a group of objects as the last-named one is a one-dimensional totality.

Now that we have sufficiently discussed the attributes that are characteristic of one and two-dimensional aggregates, we may, without any further investigation of the subject, propose the following definition, that, generally, *an n-dimensional totality of infinitely numerous things is such that the specification of n numbers is necessary and sufficient to indicate definitely any individual amid all the infinitely numerous individuals of that totality.*

Accordingly, the point-aggregate made up of the world-space which we inhabit, is a three-dimensional totality. To get our bearings in this space and to define any determinate point in it, we have simply to lay through any point which we take as our zero-point three axes at right angles to each other, one running from right to left, one backwards and forwards, and one upwards and downwards. We then join each two of these axes by a plane and are enabled thus to specify the position of every point in space by the three perpendicular distances by which the point in question is removed in a

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positive or negative sense from these three planes. It is customary to denote the numbers which are the measures of these three distances by x, *y,* and z, the positive x, positive *y,* and positive z ordinarily being reckoned in the right hand, the hitherward, and the upward directions from the origin. If now, with direct reference to this fundamental axial system, any particular specification of x, *y,* and z be made, there will, by such an operation, be cut out and isolated from the three-dimensional manifoldness of all the points of space a totality of less dimensions. If, for example, *z* is equal to seven units or measures, this is equivalent to a statement that only the two-dimensional totality of the points is meant, which constitute the plane that can be laid at right angles to the upward-passing z-axis at a distance of seven measures from the zero-point. Consequently, every imaginable equation between x, *y,* and *z* isolates and defines a two dimensional aggregate of points. If two different equations obtain between x, *y,* and *z,* two such two-dimensional totalities will be isolated from among all the points of space. But as these last must have some one-dimensional totality in common, we may say that the co-existence of two equations between *x, y,* and *z* defines a one-dimensional totality of points, that is to say a straight line, a line curved in a plane, or even, perhaps, one curved in space. It is evident from this that the introduction of the three axes of reference forms a bridge between the theory of space and the theory of equations involving three variable quantities, x, y, and z. The reason that the theory of space cannot thus be brought into connection with algebra in general, that is, with the theory of indefinitely numerous equations, but only with the algebra of three quantities, x, y, z, is simply to be sought in the fact that space, as we picture it, can have only three dimensions.

We have now only to supply a few additional examples of *n*-dimensional totalities. All particles of air are four-dimensional in magnitude when in addition to their position in space we also consider the variable densities which they assume, as they are expressed by the different heights of the barometer in the different parts of the atmosphere. Similarly, all conceivable spheres in space are four-dimensional magnitudes, for their centres form a three-dimensional

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point-aggregate, and around each centre there may be additionally conceived a one-dimensional totality of spheres, the radii of which can be expressed by every numerical magnitude from zero to infinity. Further, if we imagine a measuring-stick of invariable length to assume every conceivable position in space, the positions so obtained will constitute a five-dimensional aggregate. For, in the first place, one of the extremities of the measuring stick may be conceived to assume a position at every point of space, and this determines for one extremity alone of the stick a three-dimensional totality of positions; and secondly, as we have seen above, there proceeds from every such position of this extremity a two-dimensional totality of directions, and by conceiving the measuring-stick to be placed lengthwise in every one of these directions we shall obtain all the conceivable positions which the second extremity can assume, and consequently, the dimensions must be 3 plus 2 or 5. Finally, to find out how many dimensions the totality of all the possible positions of a square, invariable in magnitude, possesses, we first give one of its corners all conceivable positions in space, and we thus obtain three dimensions. One definite point in space now being fixed for the position of one corner of the square, we imagine drawn through this point all possible lines, and on each we lay off the length of the side of the square and thus obtain two additional dimensions. Through the point obtained for the position of the second corner of the square we must now conceive all the possible directions drawn that are perpendicular to the line thus fixed, and we must lay off once more on each of these directions the side of the square. By this last determination the dimensions are only increased by one, for only one one-dimensional totality of perpendicular directions is possible to one straight line in one of its points. Three corners of the square are now fixed and therewith the position of the fourth also is uniquely determined. Accordingly, the totality of all equal squares which differ from one another only by their position in space, constitutes a manifoldness of six dimensions.

Schubert - Part 2

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Send e-mail, comments and suggestions to

Jim Beichler, editor, *YGGDRASIL*, at

jebco1st@aol.com