CHAPTER II

MATHEMATICAL ELABORATION AND THEORETICAL DEVELOPMENT

The structure of the space-time continuum as described in General Relativity can be summarized by the equation

$$R_{ij} - (1/2)[g_{ij}R] = -kT_{ij}\ , (1)$$

where R_{ij} is the contracted Christoffel tensor or the Ricci tensor, g_{ij} is the metric tensor, R is the curvature scalar, k is the gravitational constant and T_{ij} is the energy-stress (or matter) tensor. However, this equation does not explicitly include the electromagnetic field. Many scientists have assumed that electromagnetism must play as important a role in the structuring of space-time, as does gravitation even though it does not appear naturally in the above formula. Electromagnetism can be added to the defining equation of the space-time structure in such a way that

$$R_{ij} - (1/2)[g_{ij}R] = -k[T_{ij}+E_{ij}]\ .\ (2)$$

The addition of the electromagnetic term, E_{ij}, seems rather artificial and does not lead to simple solutions for charged particles in a combined electromagnetic and gravitational field in a way similar to that of an uncharged particle in the combined field. Instead of yielding a geodesic equation, a particle of mass m and charge e in a combined field, characterized by the gravitational potential g_{jk} and the electromagnetic potential ϕ, will have a trajectory given by the equation,

$$d^2x^i + \{{}_j{}^i{}_k\}(dx^k/ds)(dx^j/ds) = (e/m)F^i{}_n(dx^n/ds)\ , (3)$$

where j, k = 1,2,3,4. This differs from the case of an uncharged particle only by the right hand side, which represents the covariant Lorentz force. In the case of an uncharged particle,

$$(e/m)\ F^i{}_n\ (dx^n/ds) \text{ goes to } 0\ (3)$$

and the remaining equation is merely the geodesic within the four-dimensional Riemannian metric.

The above equation can be regarded as a geodesic in a four-dimensional Finsler space rather than a Riemannian space. Such a Finsler space can be represented mathematically by its metric,

$$ds' = (g_{ij}dx^idx^j)^{1/2} + (e/m)\phi_i dx^i\ .\ (5)$$

This representation proves to be unsatisfactory. Each new value of e/m yields a new Finsler space, so there is no single representation for the universe with all its different particles and e/m ratios.

With the Riemannian metric[1] there are only ten independent components that can be used to represent the space-time structure. In General Relativity the ten potentials needed to describe gravitation are identified with the ten independent components of the metric tensor g_{ik}, leaving nothing to be identified with other potentials in the field such as electromagnetism. On the other hand, the electromagnetic field can be described by four additional potentials, which are equivalent to a four-vector composed of the scalar potential. In all, fourteen potentials or independent components are needed to represent the combined field.

In the equation of General Relativity, the field structure is determined by a rank two Riemannian metric tensor, which is symmetric. The number of field components for a tensor such as this is determined by the formula $N = (1/2)(n)(n+1)$, where n is the number of dimensions. Riemannian space, being four dimensional, thus yields $n = 10$ components. However, the Riemannian structure can still be saved and an adequate number of independent components found by increasing the number of dimensions to five. This alteration yields fifteen independent components, one more than is needed to describe the combined field. Kaluza was the first to see this as a possible answer to the problem of developing a unified field structure incorporating electromagnetism and gravitation on an equal footing within the geometric structure of the space-time continuum.

By introducing a space-time structure with five dimensions, Kaluza was seeking to give a geometrical representation to the generally covariant form of Maxwell's electrodynamics. Kaluza's introduction of a five-dimensional structure immediately raised two problems: (1) There are fifteen variables when a fifth coordinate is added, while the combined field of electromagnetism and gravitation only requires fourteen variables for definition; And (2) every indication implies a four-dimensional world, therefore, a five-dimensional assumption must include an explanation of the perceived absence of a fifth coordinate. These two problems are not characteristic of Kaluza's theory alone, but form the major points of contention for any theory that seeks to include more than our normal four dimensions in a physical description of the world. These same two problems also define the major lines along which Kaluza's theory has been extended by some scientists and criticized by others.

Kaluza added no physical significance[2] to his five-dimensional hypothesis, merely using it as a tool. As such, he had a great deal of latitude in overcoming both problems and was able to deal with them by assuming that the field variables[3] ($\gamma_{\mu\nu}$) were independent of the fifth coordinate (x_0) and only depended on the four coordinates of the space-time continuum when a suitable coordinate system was chosen. Mathematically, this merely means that

$$\delta\gamma_{\mu\nu}/\delta x_0 = 0 \text{ (or } \gamma_{\mu\nu,0} = 0\text{) (6)}$$

where $\mu,\nu = 0,1,2,3,4$. The component with a subscript of zero represents the fifth coordinate. This single assumption overcame both problems by allowing the space-time structure to become cylindrical in the fifth coordinate. It also had the consequence of allowing a somewhat "atrophied"[4] fifth dimension and a somewhat less generalized theory. The cylindricity would be guaranteed when a vector in the fifth dimension, A^μ, satisfied the Killing equation,

$$A_{\mu;\nu} + A_{\nu;\mu} = 0 \;.$$ [5] (7)

By further requiring that "the lines to which the A^μ are tangents - the 'A- lines' - have to be geodesics,"[6] the various A-lines in the fifth dimension were shown to be equal as well as constant, such that the norm of A was also constant throughout all of space and not only along the A-lines. This mathematical requirement or condition allowed Kaluza to base his five-dimensional structure on a special coordinate system defined by the metric

$$d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu \;. \quad (8)$$

So the new space-time structure consisted of a four-dimensional continuum that cut each of the A-lines only once (the A-lines being the five-dimensional component). The distance along any A-line could then be used to derive a value of $\gamma_{00} = +1$, thus normalizing all other components of the field. This particular result, based upon the cylindrical condition (also referred to as A-cylindricity), had the dual effect of guaranteeing that there would be no physical evidence or perception of the fifth dimension while reducing the number of variables from fifteen to fourteen, solving both of the problems which would normally have plagued any theory based upon the introduction of a fifth coordinate or dimension.

As in the General Theory of Relativity, transformations were found for this new space-time structure that were to leave the equations invariant and thus preserve the unique character of the system. This invariance under transformation was then related to the physical phenomena of electricity and electromagnetism. In particular, two transformations were found to fulfill this criterion, one being the "four-transformation" defined as

$$\bar{x}^0 = x^0$$

and $$\bar{x}^a = f^a(x^1 \ldots x^4) \quad (9)$$

where $a = 1,2,3,4$, and the "cut-transformation" defined as

$$\bar{x}^a = x^a$$

and $$\bar{x}^o = x^o + f(x^1 \ldots x^4)$$.[7] (10)

The field variables upon which these transformations act can be grouped into three general categories corresponding to the results of the transformation process; the γ_{mn}, γ_{0m} and γ_{00}. The γ_{mn} correspond in this theory to the sixteen components of a matrix representing the four-dimensional space-time continuum as in General Relativity. They reduce to ten independent components that describe gravitation. Under the four-transformation, the γ_{mn} act as a four-tensor and under the cut-transformation they are invariant. The γ_{0m} correspond to the eight components (or four + four) in a five by five matrix, which represents the mixed terms of normal space-time and the fifth dimension. Under four-transformation the γ_{0m} act as a four-vector, while under the cut-transformation they vary by an additive term, such that

$$\bar{\gamma}_{0m} = \gamma_{0m} - \delta f/\delta x^m . \quad (11)$$

It is this variation that allowed the introduction of the electromagnetic four-vector into the new space-time structure. The γ_{0m} were equated to the electromagnetic potentials ϕ_m since "this corresponds to the fact that the electromagnetic potentials are defined only up to additive terms which are gradients of an arbitrary function."[8] The final term of γ_{00}, which is purely five-dimensional, was in Kaluza's original theory set equal to +1. This term was to be found invariant and constant under both transformations and thus proved to be effectively removed from the space-time structure, as required by experience.

When these field variables were used to calculate the metric defining the space-time structure,

$$d\sigma^2 = \gamma_{\mu\nu}\, dx^\mu dx^\nu , \quad (12)$$

a new field quantity, $g_{\mu\nu}$, whose components are those of a five-dimensional tensor, was found. This resulted in the new line element

$$d\sigma^2 = (dx^0 + \phi_m dx^m) + g_{mn} dx^m dx_n , \quad (13)$$

where $\phi_m = \gamma_{0m}$ and $g_{mn} = (\gamma_{mn} - \phi_m \phi_n)$. The g_{mn} were found to correspond to the usual metrical coefficients of General Relativity, whereby the Riemannian metric is defined by

$$ds^2 = g_{mn}dx^m dx_n \,. \quad (14)$$

An antisymmetric tensor, which was left invariant, was also found to be involved in the process of transformation. This tensor was evaluated as

$$\gamma_{m,n} - \gamma_{n,m} = \delta\gamma_{k0}/\delta x^i - \delta\gamma_{i0}/\delta x^k = f_{ik} \,, \quad (15)$$

and corresponds to the electromagnetic field strength. In this manner, the dependence of the field structure on electromagnetism as well as gravitation was reflected in the metric that defined space-time and electromagnetism was wholly incorporated into the new field structure.

Once the line element $d\sigma^2$ had been found, it was easy to derive the geodesics corresponding to a physical particle's trajectory. It will be remembered that the failure of General Relativity to provide an appropriate geodesic for a charged particle in a combined electromagnetic and gravitational field was a major concern in attempting to find a unified field theory. In this respect, Kaluza's theory enjoyed a modicum of success, not found in other field theories. Two cases were found for the geodesic. The first case representing the geodesic in the fifth coordinate gave no special results, only that

$$L^2 - A^2 = \text{constant} = g_{ik}(dx^i/ds)(dx^k/ds) \,, \quad (16)$$

where $L = \delta\sigma/\delta\tau$ and $A = \text{constant} = x^o + \phi_i x^{\cdot i}$. However, for the case where the four normal coordinates were considered, the geodesic yielded

$$d^2x^k/ds^2 = \left\{ {i\,k \atop j} \right\}(dx^i/ds)(dx^j/ds) + Af^k{}_j(dx^j/ds) = 0. \quad (17)$$

When it was assumed that $A = e/m_0$, Kaluza's theory yielded the correct geodesic for a charged particle that had been sought. This assumption was justified to some extent since A is invariant and constant under transformation, as is the universal ratio of e/m_0.

The last step in the process was to derive the field equations in a vacuum from a simple variational principle. Kaluza assumed that the Lagrangian to be used in the variational principle was the product of the five-dimensional curvature scalar R and the square root of the determinant $|\gamma_{\mu\beta}|$, such that

$$\delta \int R(-\gamma)^{1/2} dx = 0 \quad (18)$$

and

$$\delta \int (\delta^i{}_n g^{kj} R_{ikj}{}^n + 1/2\, \phi_{rs}\phi^{rs}) (-g)^{1/2}\, dx = 0 \,. \quad (19)$$ [9]

This variational principle was restricted by the condition that $(\delta\phi_i)_{,0} = 0$ or rather $\gamma_{ik,0} = 0$ and γ_{00} = constant, as in the original assumption of the cylindrical condition. Under these circumstances, the field equations were found to be

$$R^{\mu\lambda} - (1/2)g^{\mu\lambda} R + (\alpha\beta^2/2)E^{\mu\lambda} = 0 \quad (20)$$

and

$$\frac{\delta[(-g)^{1/2} F^{\mu\lambda}]}{\delta x^{\mu}} = 0 . \quad (21)$$[10]

Upon the assumption that the constant $k = \alpha\beta^2/2$, these new field equations become the same as those given by General Relativity when the electromagnetic field is added from outside the theory, otherwise known as the Einstein-Maxwell equations. Therefore, Kaluza's mathematical structure of a five-dimensional space-time successfully modeled the single combined gravitational and electromagnetic field given what he was attempting to accomplish.

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