Editor's Introduction to
"Space and Its Dimensions"

This paper was originally written by Charles T. Whitmell and presented before the Cardiff (England) Naturalist's Society on the 10th of November in 1892. The copy used to reproduce the paper for this publication was found as a preprint at the British Library, London, England. The text of the paper has been duplicated as accurately as possible with the pagination of the original kept intact, so scholars may quote Whitmell's paper directly as deemed academically necessary.

It has been generally accepted that the research of William K. Clifford died with Clifford and that he had no complete physical theory of four-dimensional space, nor had he any followers. However, these historical assumptions are not born out by the facts. Whitmell was just one of Clifford's students who later publicized Clifford's ideas, although he did not attribute all of his exposition of the four-dimensional hypothesis to Clifford since many others had written on and added their own ideas to the subject in the intervening years. At least two of Clifford's students developed physical theories based on Clifford's original theory (which was never *completely* published in a finalized and logically consistent form). Karl Pearson developed a theory of 'aether squirts" and W.W.H. Franklin developed his own hyperspatial theory, while Clifford's theoretical model directly influenced the work of Felix Klein in Germany, Simon Newcomb in America, Sir Robert Ball, Charles Hinton and many others in Great Britain.

By the time that Whitmell made his presentation, twenty-three years after Clifford's tragic death, a great deal of misconceptions regarding the possible reality of a four-dimensional physical space had been propagated, very nearly spoiling any chance of conducting legitimate scientific research regarding a possible fourth dimension without undue ridicule by the scientific community. Still, there was a general curiosity with such a possibility among both scholars and the common public, equal and opposite to the skepticism of many scientists, and Whitmell's presentation was surely given to educate the public about the fourth dimension and dispel as many misconceptions as possible.

Space and its Dimensions.

(Illustrated by Models and Diagrams.)

By CHAS. T. WHITMELL, M.A., PRESIDENT.

Read on 10th November, 1892, before the Cardiff Naturalists' Society.

SYLLABUS :

The Abyss of no Dimensions - A Person of Position - The Linite, and his monotonous existence in Boredom - How he reasons in a Circle - The Filmite in his Flat - A very shallow creature - Knotty points - In Gaol - He goes to a Ball - Flirts with a Tractrix - Solid Space and our terrestrial prejudices - Reflections on distorted views - The enlightened ideas of the Tetrites of Tetratopia - Earthly Bubbles turned inside out - Oliver experiences a Twist - Weighty matters; A Message from the Stars; and a Fortuitous Concourse of Atoms - A Man whose mind expands in Space of "N" Dimensions.

INTRODUCTION.

The Paper, which I am about to read to-night, was originally intended to be delivered as a Presidential Address at the Society's Annual Meeting last month. But, as on that occasion, the necessary business of the Society did not leave available sufficient time for properly dealing with the subject, I have kindly been allowed to postpone it until now.

THE Syllabus is a very brief and imperfect summary, but I hope it may help to show that the subject is not quite so dry and uninviting as the bare title might lead many to suppose. I have tried to give a humourous aspect to the Syllabus, though the fun may perhaps appear of a somewhat elephantine character.

The following is a list of some of those to whom I am indebted in connection with this Paper. Dr. E. A. Abbott, Sir R. Ball, Mr. W. Ball, Mr. Hinton, Dr. Schofield, Mr. Spottiswoode, Mr. Willink, Professors Cayley, Chrystal, Clifford, Helmholtz, Henrici, Pearson, Poincaré, Sylvester, Tait, and Tanner. To Prof. Tanner, the accomplished Mathematical Professor of the South Wales University College, I must especially express my thanks for most kind and valuable assistance.

All of us are familiar with Space, but to define it is far from easy. Is space an innate idea, a form supplied by the mind, conditioning our perception of objects, as Kant believed; or is it purely a product of sensory experience aided perhaps by inheritance? These thorny metaphysical problems I do not propose, and am not competent, to deal with, and shall therefore simply say that, so far as I can follow the dispute, I incline to the Kantian view. In this Paper I shall take lower ground, and endeavour to bring before you some aspects and properties of space, which, though well known to those curious in such matters, are yet, I believe, not so familiar as to be merely a twice-told tale to the majority.

As Ancient Gaul was divided into three parts, so space may be considered under three heads. First, Linear space, which has only one dimension - Length. Secondly, Surface space or Area, which has two dimensions - Length and Breadth. Thirdly, Solid space or Volume, which has three dimensions - Length, Breadth, and Depth, or Thickness. A point has no dimensions, but only position. It has no extension while a line is once, a surface twice, a solid thrice, extended. The path of a moving point is a line straight or curved; that of a line is (in general) a surface flat or curved; that of a surface is (in general) a solid. The boundaries of a line are points, the boundary of a surface is a line, that of a solid is a surface.

We assume that space is continuous, that there are no breaks in it. The positions occupied by a body travelling from any one place to any other form a perfectly connected series. If they did not, either the body would cease to exist, or would be outside our space at the gaps. The Postulate of Continuity states that two adjacent portions of space have the same boundary, which is a surface, a line, or a point, in the cases respectively of two adjacent portions of a solid, a surface, or a line. Between any two points on a line there may be an infinite number of intermediate points.

Lines, surfaces, or solids, may conveniently be called Figures. Two figures which can be made to coincide are equal both in size and shape. Such figures are called congruent, and one of them is simply a duplicate of the other. Figures, alike in size, but not in shape, are called equivalent; those, alike in shape, but not in size, are called similar. A triangle and a square, equal in area, are equivalent. A large and a small triangle, whose angles are equal each to each, are similar; so also are a large and a small circle. (Fig. I.) We can illustrate change of size without change of shape by blowing bigger a spherical bubble; and change of shape without cbange of size by

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pouring a given quantity of liquid from one vessel to another of different form; or by distorting a square into a parallelogram upon the same base and between the same parallels as the square. The telescope and the microscope cause objects seen through them to appear changed in size but not in shape. The possibility of the existence of figures alike in shape, but differing in size, is characteristic of our ordinary Euclidean space.

To test the equality of two figures we endeavour to make them coincide by applying the one to the other. Lines and surfaces may be superposed. Solids may be fitted successively into the same gauge or mould. The foundation of Euclid's method consists in establishing the congruence of geometrical figures. It is assumed that figures may he moved in space without alteration in shape or size. For example, a triangle drawn in London may be taken to Edinburgh without any change in its sides or angles. Or, to put it another way, we can at Edinburgh construct a triangle in all respects equal to one in London. This implies that all parts of our space are uniform or exactly alike; that is, there are no properties in it dependent on position or direction.

NO DIMENSIONS.

Let us first consider space of no dimensions. This is represented by a Point which has position only, a distinction that it is the lifelong ambition of many of us to obtain. Imagine a being of no sensible size living in one fixed position, and let us call him a Pointite. Dr. Abbott gives us the following graphic account of such a being and his surroundings. "The realm of Pointland is the Abyss of No Dimensions. The Pointite is a being confined to the non-dimensional gulf. He is his own World, his own Universe; of any other than himself he can form no conception. He knows not length, nor breadth, nor height. He has no cognisance even of the number Two, for he is himself his One and All, being really Nothing." But Dr. Abbott forgets to say that the Pointite is a Person of Position, and surely this compensates him for many disadvantages.

UNIDIMENSIONAL SPACE. THE LINE, STRAIGHT.

Space of one dimension, linear space, next claims our attention. A line is usually said to have length but not breadth. A line is usually said to have length but not breadth. Suppose there is an endless straight cylindrical tube with a bore of no sensible breadth, and that within the tube lives a small worm-shaped being just fitting it. We will call him a Linite, and designate his tunnel-like universe as Boredom. (Fig 2.) He would be able to go only backwards and forwards. Space would seem infinite to him and alike at every part. There would be no idea of breadth (right and left), or depth (up and down), but only of lengths, longer or shorter. But these lengths would be estimated by time, not by sight; for, as only the ends of lines, i.e., points, would be visible, there could never be a side view of any line as

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a whole, or of any finite part of it. No geometrical figures would be possible in Boredom. Outside the linear world nothing would exist. If there were two Linites in the tube each could never pass the other; and even if each had an eye at each end, each would never be able to see more than one end of the other. To see a line as a line one must evidently be out of Lineland altogether and in space of at least two dimensions, and then we can see not only the ends of a Linite, but his interior as well.

Under such conditions existence would certainly be somewhat monotonous. The Linite would realise those words of Pope

> "The spider's touch, how exquisitely fine!
> Feels at each thread, and lives along the line."

A line drawn from bidimensional space will cut in only one point a line, such as we are considering, and hence the direction of this first line is not determinable by the Linite.

A curious difficulty occurs in the Linite's geometry. Let there be two equal straight lines, so placed for convenience that the second is a continuation of the first. Let the two lines he A B C, C b a (Fig. 3), C being their common point; and let b be the same distance from a that B is from A. Suppose the point C to be crimson, the points A and a to be azure, and B and b to be black, or distinguishable by some other simple means. In going from A to a the Linite would always pass the coloured points on the second line in an order different from that in which he passed the points corresponding to them on the first line. On the first line he would pass in succession azure, black, crimson; on the second, crimson, black, azure. The distance from crimson to black and that from black to azure are exactly alike in each line, but he cannot make the corresponding points coincide by such superposition as he could use, such as sliding one line alongside the other. He may, therefore, hesitate to believe that coincidence is possible, or that the lines are really equal.

But, by turning the line C b a round C in a plane, it can easily be so placed that the points a b C fall exactly on the points A B C respectively, and thus complete equality is proved. But this rotation is obviously impossible to our linear geometer, for it involves motion in a surface, that is in two dimensions. But a dweller in a plane world (one of two dimensions only) would easily perform this rotation, and thus solve a problem very perplexing to a Linite.

THE LINE - CURVED.

We pass on now to curved lines, and first to those whose curvature is in a plane. A curved line presupposes the existence of bidimensional space, for there must be a second dimension or direction in which to curve the line. But this second dimension is not perceived as such by a Linite living actually within or along the curve itself.

Suppose our Linite to live in a very narrow tube bent into a circular ring, inside which he can travel as long as he likes. (Fig. 4.) If there were nothing to distinguish one part of the tube from another, the Linite would naturally conclude that his universe was infinite; and, as at every part the curvature is the same, he would have no means of knowing that he was not in a straight tube. His body bend being constant would seem merely a factor in his physical constitution and not a peculiarity of the space he was traversing. If there were in the tube a recognisable mark, then, by noticing that it came into sight again and again, he could avoid the error of supposing his world to be infinite.

The Linite would affirm that at all parts of his universe space had exactly similar properties, and he would have for this statement grounds better than we have for affirming the same of Our world, for he would have visited every portion of his unidimensional space. The Linite's method of reasoning in a circle would be free from the errors associated with that process when attempted by illogical terrestrial beings.

It is important to notice the difference between infinite and unbounded. A circular path is unbounded, but is not infinite, although the Linite might think it was infinite.

Let us now suppose the tube bent into an elliptical shape. In such case the curvature is continually varying, with two equal minima at N N' and two equal maxima at X X'. (Figure 5.)

Let the Linite travel round in the direction P Q R S, and suppose the curvatures at these four points to be equal. He could distinguish position P from position Q, because in passing through P he is increasing, while in passing through Q he is diminishing, his bend. But he could not distinguish P from R, or Q from S, and would be liable to suppose the circuit only half as long as it really is. In stating this I assume that he knows the circuit is a finite one. By putting precisely similar marks at P and R we can secure this, and yet allow the error as to the length of the circuit to remain.

After all it is quite conceivable that the Linite might consider even his elliptical path to be one of constant curvature, and so undistinguishable from a circular or a rectilinear path. This would be the case if lie supposed the feeling, produced by changes of curvature, to be due really to some periodic alteration in his physical constitution. The Linite would be especially liable to this subjective error, if we imagine him to be at first in a straight but flexible tube, which is then by some outside power slowly bent into a circle or an ellipse.

To coil a tube into a helical shape would necessitate the use of tridimensional Space. But a Linite living in such a tube would still be conscious of space of only one dimension. A uniform helix has an Unvarying bend, and in this respect agrees with the circle, but the helix does not return into itself, so that a recognisable mark in the tube would not enable a Linite to conclude that his space was finite. In fact he would find no limit, and would be correct in supposing that there was none.

BIDIMENSIONAL SPACE - FLAT.

We may now consider the peculiarities of a world of two dimensions, length and breadth, a surface or superficial space without depth. In the first case, suppose this universe to be plane, and call it, after Dr. Abbott, Flatland. Let the inhabitants be thin, flat, filmy creatures, such as might live between two pages of a closed book, or upon the flat surface of a solid. We may term them Filmites from their shallowness, and may compare them to shadows on a surface.

A Filmite could move backward, forward, right, and left, upon his fiat, but would be unable to move up or down, or to conceive the meaning of such motions or words, for he cannot leave his plane. He would find that a moving point traces a line, and a moving line a surface, but would be quite unable to imagine the solid generated by a surface moving up or down in the third dimension.

In bidimensional space there is room for an unlimited number of spaces of one dimension side by side, just as it requires an unlimited number of lines to fill an area. And the bidimensional space though outside, as it were, any given tube of linear space placed within it, is still everywhere in contact with such line.

A Filmite would find it impossible to make a knot; for, in tying a knot, he must take some part of the cord out of the plane, and that would be out of the world, of the Filmite.

Suppose a Filmite had been committed to prison, by being put through a door in the side of a roofless square, the door being afterwards shut. Of himself he could not escape. But now endow him with superplanar, i.e., with tridimensional powers, and he could at once leap over a side of the square, and so be outside the prison wall without having passed through it. From the moment he left the plane until the moment he again alighted upon it, he would be invisible to the other Filmites, who would find his power of assuming invisibility and of escaping from gaol utterly incomprehensible. A tridimensional burglar would have a good time of it in Flatland.

Prof. Karl Pearson gives the following interesting illustration of the perplexity of a Filmite under certain conditions. Suppose an indefinitely thin flat flounder fish to live in a film of water contained between two horizontal planes. He could move only backwards and forwards or right and left, while vertical upward or downward movements would be impossible, and probably inconceivable. Make a hole in one of the planes, and squirt water in. The pressure produced by the water might compel the flounder to circumnavigate the squirt, that is, the squirt might to him be hard and impenetrable. Such squirts, though only water in motion, might correspond to what we call impenetrable matter.

If he were told that matter was formed of squirts, he would be unable to conceive where the squirt came from. The water would flow in all directions in the plane. The flounder would think us mad if we suggested the water came upwards or downwards, for he knows no

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such directions. Could the flounder get out of his space through the squirt, that is through and out in the direction of matter, he would reach a new world, wherein he would perceive what squirts were, and what his matter really was. He would reach the heaven of three-dimensioned space. But to do this is to him inconceivable and ridiculous, it would be to penetrate behind the surface of his sense-impressions.

A curious trick could be played thus. Imagine two Filmites at a great distance apart, and that we bend their plane so as to bring them almost into juxtaposition. To their astonishment the Filmites would seem close together without having stirred a single step. Even if they tried to approach each other they would find they were still far apart, for their movements could take place only along the surface, and measured along that they are no nearer than before it was bent. The result to them would be as marvellous as would be to me the appearance of a New Zealander in this room without either of us having travelled out of his own country.

If we, from our space of three dimensions, look down upon Flatland, we can see at one glance the inside of all the buildings and the interior anatomy of all the Filmites, a feat of course impossible to a Filmite, for he cannot lift himself above his plane. To us the Flatland houses are all Open, roofs being an unnecessary luxury. A line drawn from our space will intersect Flatland in only one point, and hence its direction is not determinable by the Filmites.

Whatever sort of geometrical figure is drawn in Flatland can be seen by a Filmite only as a series of lines, as he gets never anything but an edge view of it. Still, he can determine the character of a figure by measuring its sides and angles as he walks round alongside of it. To see a triangle or a square as such one must evidently be out of Flatland altogether and in space of three dimensions at least.

Dr. Abbott has suggested an ingenious condition of things, by which, even in Flatland, the shapes of regular figures could be inferred by sight alone. Suppose the atmosphere foggy, and that the Filmite looks edgeways at an equilateral triangle and at a square, these being so placed that the direct visual ray from each, bisects the angle which each turns towards him. In each case what is seen will look simply like a straight line, brightest at its middle point and dimmest at its ends; but the rate at which the shading off from the middle to the ends takes place, will, in the case of the straight line representing the triangle, be more rapid than in that of the one representing the square. Hence, with practice, and under favourable conditions, the two figures could be distinguished and determined by sight alone.

It has occurred to me that perhaps an eye, possessed of an extremely delicate sense for focal adjustment, might, even in a clear atmosphere, enable conclusions as to shape to be drawn simply from sight.

Suppose a ball, which is a solid figure of three dimensions, to alight upon Flatland, and to pass slowly down through the plane, as it would pass through a surface of water. The sections of the ball in the plane

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would always be circles. At the first contact there would be a point which, as the ball passed on, would gradually expand into a circle, growing bigger until its diameter ultimately equalled that of the ball. This circular section would then gradually contract, and when the ball had just completed its downward passage through the plane, the section would have shrunk up into a point again.

By moving along round the sections of the ball lying in his plane, the Filmite would find them to be a series of varying circles; but, having no idea of up and down, he would be unable to superpose these circles and so, as it were, construct the ball. If we suppose the Filmite stationary, all he would actually see would be a point expanding into a straight line, which would then contract into a point again. The downward (or upward) movement of the ball, to which these appearances are due, could neither be seen nor imagined by him. He could only take cognizance of what was actually in his own plane. If he could superpose the varying straight lines, which are the projections of the horizontal circular sections of the ball, he would construct a circle which would be a vertical section of the ball.

If a Filmite were translated into space of three dimensions, the flat figures, which he had previously seen as lines only, would be seen now as surfaces as we see them. How would a solid be apprehended by him under these new conditions? There can be no doubt I think that, for a long time, solids would seem to be merely flat surfaces variously shaded. A sphere would be supposed to be a circle, and a cube (viewed diagonally) to be a six-sided flat figure. Not until he had examined these solids from every side and touched them would he be able to grasp the new idea of a third dimension.

The geometry of the Filmites would be, in almost all respects, the same as our plane geometry. They would affirm that only one straight line can be drawn between two points; that such a line is the shortest possible; that the ends of a straight line can never meet however long it is; that through a point in their plane only one straight line can be drawn parallel to a given straight line; that the three angles of any plane triangle are together exactly equal to two right angles; that similar figures are possible; and that a figure can be moved along the plane without change from one position to another.

But there would be certain difficulties from which our plane geometry is free. It was shown that a Linite felt uncertain as to the equality of two particular straight lines, because he could not turn either of them round a point so as to make corresponding portions of them coincide. An analogous difficulty besets the Filmite. Suppose he meets with two triangles, like those figured (Fig. 6.), their sides and angles being equal each to each. Still he would find that, turn them round in the plane how lie would, never could he place them so that, if superposed, they would coincide. But, if he is of a reflective cast of mind, and has studied the Linite's problem, may he not conjecture that there is a method - impossible for him - by which the triangles could be made coincident? A third dimension to space might enable the difficulty to be solved.

Now we know that what must be done is to lift up one of the triangles, turn it over, and put it down again. But this process involves movement in the third dimension, and so is beyond the Filmite, who can no more rotate a figure round an axis in its plane, than can the Linite turn a line round a point.

BIDIMENSIONAL SPACE. - CURVED.

A curved surface presupposes the existence of tridimensional space, for there must be a third dimension in which to bend a surface originally flat. But this third dimension is not perceived as such by a Filmite living on the surface.

SURFACES OF POSITIVE CURVATURE.

For our first example, take the surface of a Sphere, which we know is everywhere of constant positive curvature. Let the bend of the Filmites living upon the ball be the same as that of their surface world. They could thus slide freely over it in complete contact, and without change of any kind. This unvarying character of the bend would probably prevent it being recognized at all. A Filmite would think he was still living in a plane; and, if he had been living in one, but, now that he was on a sphere, experienced a constant difference in his feelings, he might reasonably attribute such change to some alteration in his own physical constitution, and might not discover that the change was due really to his now living in space of constant positive curvature. It will be remembered that he cannot conceive a third dimension, so that the bend to him is not what it is to us, and the difference between a plane and a spherical surface, though it might suggest a bend of the nature of a plane curve, would not suggest a bend of the sort due to a curve of double curvature (or in three dimensions), such as could be traced by a Filmite if he moved over the ball in a spiral path.

A Filmite on a sphere, unless he had a mark by which he could recognise his return to the same part, might easily suppose his space to be infinite, as it was in the plane. We know that the spherical surface is finite in extent, but it is at the same time unbounded, for, there being no boundaries, a dweller upon it can go round and round for ever.

The geometry of a Filmite, dwelling on a sphere, differs remarkably from that of one living on a plane. On a sphere we of course cannot draw a straight line between two points, but we can draw a shortest line, really an arc of a great circle of the sphere. Such a line is termed a geodesic, and would be marked out by a smooth thread tightly stretched along the surface from the one point to the other. To the Filmite a geodesic will play the part of a straight line, and we may suppose the Filmites to be accomplished navigators able always to practise great-circle sailing.

Let the two points be A and B, and first let B be antipodal to A. The Filmite would find that an infinite number of routes, all equal in

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length, could be taken from A to B. As these routes are the shortest or "straightest" possible, we get, as it were, a contradiction to Euclid's axiom, that only one "straight" line can be drawn between two points. If the two points are antipodal points on a sphere, a third point is required to fix any particular great circle. Next, suppose B not diametrically opposite to A; then the Filmite would still find that there are two direct routes from A to B, but that one of these is shorter, and the other longer, than any of the equal routes travelled along before. Only in the special case of antipodal points would Euclid's axiom be incorrect.

The Filmite would also discover, if he only went far enough, that he eventually came back to his starting point. Of parallel (geodesic) lines he would know nothing; and, further, he would find that any pair of (geodesic) lines on his world would, if produced, cut each other not in one point only, but in two points, so that for him two "straight" lines could enclose a space.

Suppose the Filmite to measure the angles of a triangle, drawn on the sphere. He would find the sum of its angles always greater than two right angles, and - the bigger the area of the triangle - the greater would be the sum of its angles; the maximum, never reached, but indefinitely approachable, being six right angles, while for an infinitely small triangle the limit of the sum would be two right angles.

Again, two figures alike in shape, but different in size, that is two similar figures, cannot be drawn upon the same globe. It is true that we can draw any number of "small" circles upon a sphere, but even these are to a Filmite not similar; for their radii being measured along the surface, from pole to circle, it is easy to show that the circumferences of such circles are not, as they are in a plane, proportional to the radii. If we double the (surface) radius, we by no means double the circumference.

The foregoing peculiarites would greatly puzzle a Filmite, who had previously lived in a plane, which we may (for the sake of argument), suppose to have become a spherical surface while he was asleep. One condition, however, would remain as a consolation to remind him of his former existence in Flatland. He would still he able to move from one part of the surface to another, without alteration either in shape or size, any figure drawn upon it. This property of the surface could be discovered you see by Filmites quite ignorant of a third dimension, for it is absolutely independent of any points outside the surface.

The original Euclidean geometry familiar to a Filmite, when resident in a plane, would be none the less true, but would not apply to an ideal space, and not to that of his present experience. The new geometry, associated with a sphere, would not be attributed by Filmites, who had lived in a plane, to a bending of the plane, for this would imply, as already pointed out, a third dimension, in which the bending could take place. But, if anything led them to conceive a third dimension, they might elaborate a science, such as our solid geometry only to them it would deal with imaginary objects in an imaginary space.

Those Filmites, who were given to speculation, would often be in a state of divine discontent, and be constantly seeking for a solution of the problem: Why is our space limited to two dimensions?

As another example of constant positive Curvature we have the surface of the Spindle, produced by rolling up into a closed figure a hollow hemisphere (Fig.7). This surface has everywhere constant curvature, because, at any point of it, the decrease of curvature along any meridian passing through the point, as we travel from the equator to the pole, is exactly compensated by the increase of curvature in the arc of the normal section perpendicular to the meridian at the point. Thus the curvature of the surface of a spindle resembles that of the sphere in being positive and constant; but the curvature of a geodesic on a sphere is uniform at every point and equal to that of any other geodesic on the sphere, and this is of course not true of the spindle's geodesics.

The geometry of the dwellers upon a spindle would resemble very closely that of the dwellers upon a sphere; and the former, like the latter, could move, without alteration, from one part of the surface to another, any figure drawn upon it. This means that the angles and lines of a flexible figure, measured on the surface itself, would not be changed by the movement. There would be variable bending of the lines in the third dimension in the case of a spindle, which would not be required in the case of a sphere, but this bending would not alter the measurements, and would not be recognisable by the

Filmites. A flexible figure would fit exactly at every part without any extension or contraction, and so need not be elastic in its nature.

It may be of interest to refer very briefly to the geometry associated with living on a surface, not of constant, but of varying, positive curvature, such as the surface of an Ellipsoid, or an Ovoid. On such a surface a figure could not be moved about without changing its size (the lengths of its lines), and its shape (the magnitude of its angles). These would not be independent of position. A triangle drawn on the earth at the equator would be altered if transferred to either pole. A Filmite, to make his body fit against an ellipsoid, would require not only to bend it differently at different parts, but also to stretch (or contract) it according to his position. These bendings and stretchings would go on continually as he moved, and upon a symmetrical surface, like an ellipsoid, two places of equal curvature might be distinguished from, and two other places of equal curvature might be confused with, one another, in a way analogous to that described in speaking of the Linite in an elliptical tube. On a surface of irregular curvature, on the other hand, the Filmite might determine his position by the degree to which his body was bent and stretched. Yet it is quite possible that he would think he still lived in a plane, and that his feelings of continual change arose from his physical constitution, and were quite independent of the geometrical character of his space.

Suppose the Filmite lived on an ovoid or egg-shaped surface, and that he drew a triangle and a circle at each end, the triangles having

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their (geodesic) sides equal each to each, and the circles their (geodesic) radii equal. Then he would find that the angles of the little-end triangle would exceed those of the big-end one, but that the circumference of the little-end circle would fall short of that of the big-end one. Dean Swift has told us how seriously the Big-endians and the Little-endians differed both in politics and religion. We learn now that they differed in their geometry also.

SURFACES OF NEGATIVE CURVATURE.

A surface of negative curvature has its principal curvatures turned opposite ways, so that the one is convex and the other concave. The surface of a saddle, and that of the inside of a ring, are good examples. The negative, like the positive curvature, may either be variable or constant. If variable, the surfaces are, as it were, counterparts to the surface of the ellipsoid, and to that of other figures of variable positive curvature. If constant, the negative surfaces are, as it were, counterparts to the surface of the sphere, and to that of other figures of constant positive curvature. A surface of constant negative curvature may therefore be called a pseudo-spherical surface. Its disconnected character, opposed inclinations, and utter disregard of Euclid, might suggest to sarcastic people that we were dealing with a feminine surface, a sort of superficial female.

One type of this surface may be obtained by spinning round on its asymptote as axis, the curve known as the Tractrix. Miss Tractrix as a spinster thus tends to confirm the view of our sarcastic friends, and we shall shortly consider the opinions of a Filmite who is bold enough to flirt with her. This tractrix-type is shown in (Fig. 8a), and resembles somewhat the stem of a long glass vase. It may be called a Tractroid.

Another type of pseudo-spherical surface is shown in (Fig. ab), and may be produced by spinning the curve A C B round the axis P Q. The figure somewhat resembles a hyperboloid of revolution, but differs in the fact that the curvature of the meridian increases from C to A, or C to B, instead of diminishing, as it does in the hyperboloid. Meridional strips of this second type are somewhat similar to those of the inner surface of a ring, but are of constant instead of varying negative curvature.

The plaster models shown I procured from Brill of Darmstadt. I doubt very much if such things are obtainable in England, where we are still far behind Germany in general appreciation of science.

Of surfaces of constant positive curvature there are three types, and corresponding to these are three types of surfaces of constant negative curvature. The spindle type corresponds to a conical type, a bolster type to the hyperboloidal type, and - intermediate to these pairs - the sphere corresponds to the tractroid.

On surfaces of constant negative, just as on those of constant positive curvature, flexible figures may be moved about without alteration of either size or shape. The Filmite would not need to stretch or contract himself, but simply to bend his body in the unrecognisable

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third dimension, as he moved about. The bending alone would enable him always to fit the surface exactly at all parts of it. His body must be flexible, but need not be extensible. In other words, he can construct on any one part of the surface a figure whose lines and angles shall be equal to those of a figure on any other part of it, the measurements of the two figures being of course made along the surface. The flexures in the third dimension may differ, but, on a pseudo-spherical surface, these would not affect the surface measurements.

Between two points on opposite meridians two equal shortest lines could be drawn; but, with this exception, the Filmite would find that, on a pseudosphere, as on a plane, only one shortest line could be drawn between two given points, and also that this line, or geodesic, however far it is produced, never returns upon itself.

Understanding by line, a geodesic, it will be remembered that on a sphere no line can be drawn through a point parallel to a given line, and that on a plane only one such line can be so drawn. But on a pseudosphere a whole pencil of lines can be drawn through a point, no one of which, however long it is, ever cuts a given line. This pencil is limited by two lines, one of which intersects one end of the given line at an infinite distance, and the other of which similarly intersects the other end. These two lines are the two parallels characteristic of the surface. The other lines in the pencil may be termed non-intersectors.

If the Filmite constructs a geodesic triangle on his pseudosphere, and then measures the angles, he will find their sum always less than two right angles, and the larger the area of the triangle the less will be the sum of its angles. A triangle infinitely large would have the sum of its angles equal to zero, one infinitely small would have the sum equal to two right angles.

Lastly, two figures, alike in shape, but different in size, that is two similar figures, cannot be drawn upon a pseudosphere, the plane being the only surface upon which such figures can exist.

We learn then that, it a surface admits of the figures on it being moved about unaltered, such a property is a special one, belonging only to surfaces of constant positive or negative curvature, and to the plane (which has zero curvature) with its rolled-up forms the conical and the cylindrical surface.

We see also the differences and similarities that would be found in the geometries evolved by beings living on these surfaces. Only in the case of the plane would Euclid's axioms be universally true. Through a point on a plane only one line (geodesic), on a sphere no line, on a pseudosphere two lines, can be drawn, parallel to a given line on the surface. Between two points on a plane only one shortest line; on a sphere one line, or an infinite number of equal lines; on a pseudosphere only one line, or at most two equal lines, can be drawn. The angles of a triangle are equal to, greater than, or less than, two right angles, in the same three cases. Only on a plane can figures, alike

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in shape, but different in size, be drawn; so that the possible existence of similar figures is a simple test for distinguishing Euclidean from non Euclidean geometry.

It should be noted that if we take a sufficiently small part either of a positively or negatively curved surface, the geometry may not perceptibly differ from that of a plane so far as our observations serve, and the less the curvature of the surface, the larger may be the portion of it which is for practical purposes plane. On the earth's surface the area of a spherical triangle, whose sides are each 100 miles, exceeds the area of the corresponding plane triangle by less than one-ten thousandth part of the plane triangle.

It follows that what Professor Chrystal calls a Micranthrope, or microscopic man, though living on a curved surface, might never detect that it was not plane, provided the curvature were very slight. On the other hand, a Macranthrope, or giant, would have a geometry such as we have described in speaking of curved surfaces.

There has been suggested a system of geometry, in which the measuring rod is supposed to shorten or shrink up more and more as it recedes from a certain fixed central point in a plane. If this shortening be rapid enough, we may find that the distance of a point, really situated at a finite distance from the centre, becomes infinite, when measured by the rod; no finite number of repetitions of the ever-shrinking rod being sufficient to cover the given distance. Therefore, beyond a certain area surrounding the centre, there will be another area, of which no point can be reached. This area, in Professor Cayley's words, will be an unknowable land, an imaginary space.

Lobatchewsky invented a system of geometry, by denying Euclid's axiom of parallels. He supposes that through a point two lines can lx drawn parallel to a given line. Retaining all the other axioms, he evolves a system, which turns out to be essentially the same as Beltrami's pseudo-spherical geometry; and both systems also agree with that worked out on the supposition of the use of the contractile measuring rod, a system due to Klein, who has also, by other properly chosen laws of measurement, evolved systems corresponding to plane and to spherical geometry. Klein's systems of geometry, analogous to those of plane, spherical, and pseudo-spherical surfaces, have been called homaloidal, elliptic, and hyperbolic, respectively.

Euclid's celebrated 12th axiom about parallels cannot be demonstrated, and yet the assertion made in it is equivalent to the assumption that the sum of the angles of a plane triangle equals exactly two right angles. Lobatchewsky appeals to have been the first who dispense with the axiom and evolved a system of geometry independent of it, system in which the three angles of a triangle are less than two right angles, that is the system belonging to the surface of a pseudosphere. Clifford praises the Russian geometer very highly, saying that he was to Euclid what Copernicus was to Ptolemy. As our conception of the Cosmos was changed by Copernicus, so our conception of Space was changed by Lobatchewsky. Euclidean geometry is practically true

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within our limits of space and time; it is true here and now; but it may not be true outside those limits in the range of there and then.

PARADROMIC RINGS.

I conclude my remarks upon surface geometry with a brief reference to what are called Paradromic Rings. Take a strip of paper, bend it round into a ring and gum the ends together. We get thus a short hollow cylinder, or ring, with two surfaces (the inside and the outside), and two edges. Now take another strip, and give the paper a half twist before gumming the ends together. We thus produce a twisted ring with only one surface and only one edge. If the flat strip was 10 inches long, we shall find that the single edge of the twisted ring is 20 inches long, and that we can draw along its surface a continuous line of the same length, the end of such a line coinciding with its beginning. Let the line be drawn along the middle of the strip.

When we have travelled halfway we shall have reached a point exactly opposite (through the paper) to the starting point, and yet we have not crossed the edge of the paper. This Curious result might have seemed impossible, for we cannot get to the other side of a flat sheet of paper unless we go round the edge of it. But the half twist reduces the two surfaces and edges into one of each. When we have thus travelled 10 inches we are as far as we can be from the starting point, and yet in another sense we are almost in contact with it. For, if the paper be considered to be a surface only, we have only surface between the two points. Move on 10 inches further and we reach once more the starting point.

A remarkable geometry, that of single elliptic space, is suggested by the twisted ring, but I cannot enter upon it here, nor can it really be illustrated by diagrams or models. In this space, every shortest or straightest line is re-entrant, as it is on a sphere, and yet two such lines intersect in only one point. A complete plane in this space is finite, not infinite, and its "two sides" are not distinct, as we can pass continuously from a point on one side of it to the same point on what is called the "other" side of it, that is to say we do not need to go over an edge. Quite apart from single elliptic space, the subject of paradromic rings, the results obtained by twisting and cutting them, and the links and knots they exhibit, are very fascinating.

TRIDIMENSIONAL SPACE.

Ordinary solid space, in which we live and move around have our being, has three dimensions, length, breadth, and depth. Such a space is traced out by moving a surface at right angles to itself. In this space there is room for an unlimited number of bidimensional spaces, and for a doubly unlimited number of spaces of one dimension; and tridimensional space, though outside, as it were, any surface placed within it, is yet everywhere in contact with it.

We assume that this space is everywhere alike, for solid bodies do not alter their size or shape when moved about in it; or, if they change, we attribute the alteration to a physical cause, such as heat. It would be curious to speculate upon the nature of the geometrical ideas evolved in a world entirely composed of fluids, for a fluid can alter both in shape and volume when it is moved about. It is not at first easy to imagine a kind of space, in which the size and shape of solid bodies vary with their position; but we shall be helped if we consider what we see in a mirror. In a plane mirror the image appears of the same shape and 'size as the object, no matter how the latter is moved. In such a mirror, if lateral inversion be neglected, we have a correct representation of ordinary space. If there are at different distances several objects their images will appear to be separated by distances corresponding exactly with those between the objects.

But in a curved mirror the reflected picture represents a space in which form and size are dependent upon and therefore vary with position. Suppose the mirror to be convex and spherical, the radius of the sphere being two inches. In such a mirror a very distant object would have its image at the principal focus, that is one inch behind the surface. Between this image and the surface would lie the images of all less distant objects. All the images will be reduced in area, and will also be flattened. The flattening along a radius drawn to any object, that is to say, the reduction in the thickness or third dimension, is relatively greater than the decrease in the surface dimensions, that is in the length and breadth measured in a plane at right angles to the radius. In such a mirror the images of near objects are seriously distorted. Straight lines appear curved, and curved lines may appear straight.

In a spherical mirror, which I possess, the image of a cube, held close to and with a corner towards the mirror, looks exactly like the image of a tetrahedron in a plane mirror. On the other hand, we may so distort an object that its image appears like the object before distortion. The reflection rectifies the distortion.

Let us suppose the images behind the surface of the mirror to constitute a real world of solids, varying in size and shape as they move about; then we see that as the measuring instruments used in such a world would similarly vary as they moved, the inhabitants could not recognise the variations, but would think they were dealing with ordinary rigid bodies. If the Mirrorites could look out upon our world, they would suppose it to be a picture in a spherical mirror, and would talk of us just as we have talked of them. Nor is it easy to see how we could convince them that ours was the true and theirs the distorted view. The reader will be reminded of "Alice through the Looking-glass."

All the well-known methods of map-projection involve more or less distortion, for they are attempts to represent upon a plane the undevelopable spheroidal surface of the earth. In Mercator's Projection the globe is treated as if it were a cylinder, which has been rolled out flat.

The equatorial portion is pretty correct, but the polar regions are enormously distended.

Besides assuming that our space is everywhere alike, we also assume the possibility of the real existence in it of two planes, or of two lines in the same plane, which planes or lines shall be parallel, that is shall never meet, however far produced. This second assumption implies that our space is an uncurved tridimensional space, and not such a solid space curved in the fourth dimension; that it is, in fact, analogous to the plane, not to the curved, forms of the bidimensional space already considered. All astronomical and terrestrial measurements appear to justify this assumption.

We cannot expect to be able to test very accurately the possibility of the continued parallelism through great distances of two lines or planes. But we know that the sum of the angles of a triangle is exactly equal to two right angles only in the case of a plane triangle, and that in a spherical or pseudo-spherical triangle the sum is greater or less than two right angles, the difference increasing with the area. Supposing the space curvature to be but small, it may be impossible to detect it in triangles whose sides can actually be measured. Let us therefore take the severest test available, and investigate those gigantic triangles, whose apices are the fixed stars and whose base is the diameter of the earth's orbit. The investigations of annual stellar parallax are just those needed to test whether any of these mighty triangles has the sum of its angles unequal to two right angles; There is no reason to think that any measurable difference has ever been detected. We find nothing to invalidate the assumption that our space is uncurved. Or, rather, if our space is curved, its curvature must be inappreciably small compared with the size of the solar system.

If our space were positively curved, the parallax of a star would be greater, if negatively curved smaller, than it is in uncurved space. In pseudo-spherical space even infinitely distant stars would show a parallax, but it would be a negative one. If a negative parallax were really observed, space would be proved pseudo-spherical. The statements that such a parallax has been observed have hitherto always been traceable to errors in the observer. Yet, though we have no evidence of the curvature of our solid space, we must not, under the influence of terrestrial prejudices, extend to the whole of space the assumption that holds true for the extremely small portion known to us. In this matter it is our duty to remain agnostics, and to say - We do not know. The visible universe, immense as it is, may be but as an atom in relation to the totality of space.

It may be of interest here to point out that there is no means known to us by which the heat radiated from the sun and stars can be returned to them. So far as we can tell it is lost to them forever. A curvature in space has been suggested as affording a condition under which this radiant energy might be restored to its sources.

If space is uncurved it is infinite in extent, if positively curved it is finite, and, as it were, returns into itself, like the circumference of a circle or the surface of a sphere.

Just as a Linite in a circle, or a Filmite on a sphere, was unable to form a geometrical idea of the extra dimension, the second or the third respectively, in which the line or the area was curved; so, in the same way, are we unable to form an idea of the curvature (if any) of our tridimensional space. To visualise this we should need to view our space from space of dimensions higher than three, for the curvature of a tridimensional body presupposes a fourth dimension in which to bend it. I Am thus naturally led to the next part of my subject.

QUADRIDIMENSIONAL SPACE.

May there not be space of four, or of even higher dimensions? A moving point traces a line, which is unidimensional; a line moving at right angles to itself marks out a surface, which is bidimensional; a surface moving at right angles to itself generates a solid, or tridimensional space.

But if we move a solid we do not get anything new, for there is no new independent direction left in which to move it. When a line or a surface is moved, as described, the line or the surface, in each of its new positions, contains no point that was in the line or in the surface in any one of its former positions. But this is not so for the points of a moving solid. We reach then a limit. There are three and only three directions, each direction at right angles to both the others and independent of them. Space, as we know it, has only three dimensions. But if we could continue the process, and move a solid at right angles to itself, we should produce a four dimensional figure, occupying space of four dimensions, which perhaps we may venture to name Tetratopia. It is like Utopia, only more (Sir Thomas More) so.

True we cannot conceive this new fourth direction, in which the solid must be moved, but then the Linite and the Filmite were equally unable to conceive the direction of the movement by which a surface, or a solid, respectively, could be produced. For simplicity assume that we can move a cube in the unknown fourth dimension, that is in a direction at right angles to the three mutually perpendicular directions known to us. The ideal figure so produced is called by Mr. Hinton a Foursquare, and he concludes, from analogy, that it would have 16 points, 32 edges, and 24 surfaces, and that it would be bounded by 8 cubes. For, just as points bound lines, lines surfaces, and surfaces solids, so solids form the boundaries of quadridimensional figures.

How would a Foursquare appear to us in our tridimensional space? We know that in Flatland a cube could be recognised only in section, that is as a square. So it seems that a Foursquare would appear to us as a cube, that being the shape of its section by our space. Suppose a quadridimensional figure to come from Tetratopia, pass through our space, and vanish again into Tetratopia. We should recognise the figure sectionally in transit as a varying solid body; and to form a proper idea of it, we should have to weld into unity the series of solid figures which are the sections of it by our space; just as in the case of the Filmite and the sphere which passed through his Flatland.

By perspective we can, upon a flat surface, represent a solid (Fig. 9), the only thing wanting being the actual content of the solid. The third dimension, though suggested on the canvas, exists only in the artist's mind. Similarly a four-dimensional figure would perspectively be projected into our space as a solid, wanting of course the four-dimensional content.

Mr. Hinton helps us to realise the matter thus: Let two cubes be placed somewhat diagonally, but with their sides parallel. If we then join by lines all their corresponding points, we get a set of solid figures, eight in number, representing somewhat distortedly the bounding cubes of a Foursquare, and every plane and line in the Foursquare will be represented in a kind of solid perspective. The true content is of course absent. I exhibit a rough model of what has just been described.

Suspend a small skeleton cube symmetrically within a large one by joining together by thin wires the eight pairs of adjacent corners. The model thus constructed is a projection or perspective representation in three-dimensional space of a Foursquare. The model comprises an outer and an inner cube and six distorted cubes (prismoids), making in all eight boundary cubes. The case is exactly analogous to the projection of a cube upon a plane. The cube is represented by two squares and four distorted squares, making in all six boundary squares.

The following brief summary shows how by analogy we infer the peculiarities of a Foursquare: -

I. In one dimension we have straight lines. Take a line 3 inches long. This may be formed by a point moving through 3 linear inches. Such a line has two points.

II. Two dimensions may he represented by a square, and formed by moving a 3-inch line through 3 inches in a direction perpendicular to itself. This square has 4 points (at the angles), 4 lines (its sides), and I face, whose area is 3^2, or 9 square inches.

III. Three dimensions may be represented by a cube, and formed by the square moving perpendicularly through 3 inches. This cube has 8 points (at the solid angles), 12 lines (the edges), and 6 squares (the faces), and its volume is 3^3, or 27 cubic inches.

IV. Four dimensions may he represented by the figure generated by moving this cube through 3 inches in the fourth direction. The resulting figure, a Foursquare, has, by analogy, 16 points, 32 lines, 24 plane faces, and for its boundaries 8 cubes. It contains 3^4, or 81 "Foursquare" inches.

The following Table sums up these results

		1st Dimension	2nd Dimension	3rd Dimension	4th Dimension
Points...	...	2	4	8	26
Lines ...	...	1	4	12	32
Surfaces	...	0	1	6	24
Boundaries	...	2 points	4 lines	6 squares	8 cubes
Contents	...	3 lin. inches	9 sq. inches	27 cubic ins.	81 4-sq. ins.

In developing the Foursquare from the cube, Mr. Hinton reasons as follows: Each of the 8 points of the cube traces a line, so we get 16 points. Each of the 12 lines of the cube is doubled or repeated, and, adding the previous 8 lines, we get 32 lines. Each of the 6 surfaces of the cube is repeated, and each of its 2 lines traces a surface, so we get 24 surfaces. Each surface of the cube produces a cube, and the first and final positions of the cube give 2 cubes more, or 8 in all. The cube-in-cube model shows these details very clearly.

A similar line of reasoning tells us that, if a Foursquare be moved in space of five dimensions, the five-dimensional figure would possess 32 points, 80 lines, 80 faces, and 40 cubes, and would be bounded by 10 Foursquares, or Tessaracts. In space of seven dimensions the resulting figure would possess 128 points, 448 lines, 672 faces, 560 cubes, 280 tessaracts, and 84 five-dimensional figures, and would be bounded by 14 six-dimensional figures. These details are from a paper by T. P. Hall, an American Mathematician.

In algebraical language the boundary of a solid (say a sphere) is represented by a function of x, y, z, the co-ordinates of any point in its surface. It we put z = 0, we may obtain a series of parallel plane sections of the body (circles in this case), and then get an idea of its shape by mentally superposing these sections. In four dimensions the boundary of a body would be represented by a function of x, y, z, w, the four co-ordinates of any point in its surface. By putting w=o, we get a series of parallel sections, which would be solids, and by mentally combining these we attempt to form an idea of the shape of the body.

Quadridimensional space will have room within it for an unlimited number of spaces of three dimensions, and though, as it were, outside solid space, yet is everywhere in contact with it.

Let us for convenience, and with many apologies to classical scholars, call the imaginary inhabitants of our imaginary Tetratopia by the name Tetrites, and endeavour to appreciate some of their enlightened and far-reaching ideas. Just as we can see and touch every point of a figure in a space of two dimensions without having first to pass through any other point in such figure, so a Tetrite can see and touch every point of a solid in our space without having first to pass through any other point of the solid. So it appears from analogy that a line drawn

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from Tetratopia into our space would pass through only one point in it; and hence that we could not determine the direction of the line, for two points are necessary to determine a line.

The insides of all our buildings and the interior anatomy of our bodily frames would clearly all be open to a Being who could view us from the fourth direction. Let us imprison a Tetrite by putting him into a completely closed box in our space; then to escape, he would simply have to pass out in the fourth dimension, then move a little, and on coming back he would be outside the box. None of Her Majesty's Prisons could retain a Being who could slope in the fourth direction.

To a Tetrite

> "Stone walls do not a prison make,
> Nor iron bars a cage."

This escape of a Tetrite from one of our tridimensional gaols is of course analogous to the escape of one of ourselves from a Flatland prison by making use of the third dimension. We can reach the other side of a line by leaping over it at any part, but a Filmite must go round the end of it, and so cannot escape from a closed line.

If the spiritual part of our nature be assumed to possess quadridimensional powers, we see how we might explain otherwise inexplicable phenomena. The vision, related by St. Paul, in the opening verses of the 12th Chapter of the 2nd Epistle to the Corinthians, and the 95th Stanza of Tennyson's "In Memoriam," beginning "By night we lingered on the lawn," may perhaps be referred to here. It has even been suggested that Heaven may exist in Tetratopia, and that the birth, growth, life, and death of animals and men may be explained as the passage of finite quadridimensional bodies through our tridimensional space. Clairvoyance, hypnotism, telepathy, thought-reading, psychometry, etc., are by some people believed to be connected with the hypothesis of a fourth dimension.

In dealing with space of one dimension, or of two dimensions, it may be argued that the Linite and the Filmite, having no thickness, are to us mere abstractions. But we may make them into realities by endowing them with a uniform but very small thickness. Continuing the parallel, we, if we occupy space of only three dimensions, must be to the Tetrites mere geometrical abstractions. This difficulty may be overcome by supposing that we have a uniform but very small thickness in the fourth dimension, in which case we are living in Tetratopia without knowing it. This small thickness may, perhaps, be comparable with the dimensions of a molecule. If there is this thickness, or rather thinness, our world will constitute a sort of molecular film to the Tetrites, just as Flatland seems to be to us. But by the Idealist Philosopher, both of our space and of space of higher dimensions, perhaps the theory - that Beings exist only in the mind of the Thinker

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who conceives them - will be preferred. There will be grand scope for a Berkeleian Tetrite, and the ancient aphorisms: What is mind? Never matter; What is matter? Never mind; will receive a fresh development.

But even in Tetratopia there would be some slight drawbacks. For, as in Flatland, knots would be impossible, and, to ladies at least, this appears embarrassing. On the other hand, the young Tetrites would be able to amuse themselves by turning inside out, without stretching or tearing, such playthings as India-rubber balls, soap bubbles, or closed shells of any flexible material.

If we look at the skeleton outline of a cube (Fig. 10), perspectively represented upon paper, we can, by a mental effort, cause the front and the back faces of the cube to change places, while, at the same time, the two outer faces go inside and the two inner ones outside. Is there an analogy between this process and that by which the Tetrites turn hollow bodies inside out?

There are some curious difficulties in our tridimensional geometry. You will remember that the Filmite, because he could not use the third dimension, was unable to bring into coincidence two given plane triangles, although they were equal in all respects, the one being in fact the looking-glass image of the other. Now we can construct on the surface of a sphere two triangles, whose sides and angles are equal, each to each, but the triangles cannot be so superposed as to coincide (Fig. 11). Even if one of the triangles is removed and turned over, the opposed curvatures in the third dimension prevent superposition. To make the removed triangle fit we must not only turn it over, but must turn it inside out, or reverse its curvature, as well.

We may also have two screws or helices, the one right the other left handed, equal in pitch diameter and length, but the one cannot be made to occupy exactly the same portion of space as the other can occupy. Or, we may have two solids, such as our right and our left hand, equal in all respects, part for part, and yet the one will not fit into a would made for the other. It seems perfectly reasonable to suppose that for a Tetrite these difficulties would not exist. In the case of each of these pairs, he would rotate one of the members round a plane into Tetratopia, and then, on bringing it back, it would be in such a condition as to be undistinguishable in shape from its fellow, and could be placed so as to occupy exactly the space vacated by the latter. But rotation round a plane is a sort of twist inconceivable by us. We are limited to rotation round a line, or axis, just as is the Filmite to rotation round a point, while the Linite cannot accomplish even this. Little Oliver, despite his Surname, was not a dweller in Tetratopia.

If we stand in front of a plane mirror, the image of our left hand looks like a right hand. But, if we pass to the back of the mirror, we shall find that our real right hand would occupy the place at which we saw the virtual image of our left hand. Now, in Tetratopia, we should simply twist our left hand round a plane, and thus make it appear like a right hand, that is like the lateral inversion of our left hand in a

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mirror. Similar remarks apply to the two helices, for the image of a left-handed helix is right-handed, and vice versa.

A left-hand glove, turned inside out, becomes a right-hand one, but is then not suitable for wearing, for the wrong side is exposed. In Tetratopia, by rotation round a plane, we could make the left-hand glove fit the right hand without exposing the wrong side of the leather. We could do in fact what a mirror does in appearance only. If we could pull our left hand through itself, as we can pull a glove, we should change it into a right hand; and, if the hand were gloved, the glove would of course be changed along with the hand: similarly, if one of the triangles (Fig. 6) be pulled through itself, it will become like the other.

RELATIONS BETWEEN SPACES DIFFERING BY ONE DIMENSION.

It may be useful very briefly to summarise the relations between A and B, when A inhabits Space of one dimension higher than that in which B dwells. With some modifications I follow Dr. Schofield's scheme. A can enter or leave B's world at will and without altering his own shape. A can come as near as he likes to B without being seen, so long as B's world is not entered. A can see the inside of everything in B's world. When A enters B's world, only a part of him is visible, and such part is seen always subject to the lower dimension in which B lives. It would seem that B can be only an abstraction to A.

To B all conception of A's world seems impossible, as it involves the idea of a higher dimension. It this is so, such a world is to B non existent. While in his own world B cannot see the full and true shape of bodies which are of the same dimensions as his world, but only that of bodies of lower dimensions. Fully and properly to see the former bodies B must be translated to A's world. To B, thus translated, A would at first be seen as if possessed only of the dimensions of the world to which B really belongs; but by careful investigation the difference could be discovered.

In space of one dimension only points could be seen; in space of two dimensions only lines; in space of three only surfaces. If we use two eyes, the two slightly different aspects of the surfaces of a solid are mentally interpreted so as to suggest solidity. But we do not really see the third dimension, or depth, for to do so we should need to view the back of an object through, and at the same time as, the front of it. We can, of course, go round and see the back, but that is not the same thing as seeing both back and front simultaneously.

In four dimensions, it seems, from analogy, that we should see a solid of three dimensions really as such, but should not actually see the fourth dimension. We could discover that by touch and movement. It should be home in mind that not Distance, but Direction, determines the character of a space. However long a line is, it fills only one dimension; however large a surface or a solid, we still have space of only two or three dimensions. The new or fourth direction is not at an infinite distance away from us. We may actually be living in it.

A world of any given dimensions includes, besides its own figures, all those capable of existing in worlds of lower dimensions.

VIEWS OF SPOTTISWOODE, CLIFFORD, TAIT, AND YOUNG.

The late Mr. Spottiswoode writes of four-fold space:

"It is like a rainbow, which if we try to grasp it, eludes our touch, but still, like a rainbow, arises out of real conditions of tangible qualities, and serves a definite purpose in mathematical science."

Again, he writes:

"When a space, already filled with material substances, is mentally peopled with immaterial beings, the imagination seems to have added a new capacity, a fourth dimension, of which there is no experimental evidence."

The late Professor Clifford has speculated upon the possibility of inferring, from certain unexplained phenomena in Light and Electromagnetism, that our three-fold space is in the act of undergoing, in space of four dimensions, a distortion analogous to that which would perplex a Filmite were his originally flat universe to be crumpled or bent in the third dimension by some external force. Clifford supposed that this distortion may be what really constitutes what we call motion, whether it be the motion of matter or of ether. On this hypothesis space-twist is exactly analogous to magnetic induction. The most general hypothesis appears to be that space may vary in its properties at different parts, and that at these parts the properties may vary with the time. Clifford also imagined that matter might be a wrinkle in our space, the wrinkle being bent in four-fold space. If we suppose the Flatland of a Filmite to be wrinkled at some part, but that the Filmite was rigidly flat, then evidently he could not fit the wrinkle. It would, like matter, be impenetrable, and either he or the wrinkle would have to get out of the way.

When the problem to be solved is a complex one, geometrical methods are admittedly far more difficult than algebraical ones. I remember, when at Cambridge, Professor Clifford told me that, after following out geometrically up to a certain stage some difficult problem, he was often compelled to fall back upon analysis in order to complete the solution. This remark of a man, said to have been the greatest Geometrician since Newton, seems worthy of record here.

It is stated that the great German Mathematician, Gauss, laid aside several problems, which he had treated analytically (i.e. by algebraical methods), observing that he hoped to apply to them geometrical methods in a future state of existence, when his conceptions of space should have been amplified and extended.

Professor Tait, of Edinburgh, one of the most distinguished Natural Philosophers of the day, writes as follows :-

"The properties of ordinary space involving, we know not why, the essential element of three dimensions, have been investigated by Riemann, Helmholtz, and others, and the result of their inquiries leaves

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it yet undecided whether space may or may not have precisely the same properties throughout the universe. An inhabitant, living in a sheet of paper, and incapable of appreciating the third dimension, would feel, if his sheet were crumpled, some difference of sensation in passing from portions of his space which were less to those that were more curved. So it is possible that, in the rapid march of the solar system through space (a march estimated at some half million miles a day towards the constellation Hercules), we may be passing into regions, where space has not precisely the same properties as it has here; where, in fact, we may find something in three dimensions analogous to crumpling in two dimensions, something which will imply a fourth dimensional change of form. Just as points are the terminations of lines, lines the boundaries of surfaces, and surfaces the boundaries of solids, so we may suppose that our tridimensional matter may be the mere skin or boundary of an Unseen Universe, whose matter has four dimensions."

The celebrated Dr. Thomas Young, reasoning from the consideration of - first, ordinary matter, then porosity, then the ultimate atoms, then the ethereal medium connected with heat, light, and electricity, then the unknown medium or cause to which gravitation is due, expresses the following views: -

"It seems natural that the analogy may be continued still further, until it rises into existences absolutely immaterial and spiritual. We know not but that thousands of spiritual worlds may exist unseen for ever by human eyes; nor have we any reason to suppose that even the presence of matter, in a given spot, necessarily excludes these existences from it. We may speculate with freedom on the possibility of independent worlds; some existing in different parts of space, others pervading each other, unseen and unknown, in the same space, and others again to which space may not be a necessary mode of existence."

I should like to add to this remarkable passage two lines from a modern poet: -

"Star to star vibrates light: may soul to soul
Strike through a finer element of her own?"
-TENNYSON.

The great Principle of Continuity forbids us to set a limit either to matter or spirit. But personally I incline to the belief that the one differs from the other not in degree but in kind.

Here may be mentioned, though I do not attempt to draw any conclusions from it, what Mr. Crookes calls the ultra-gaseous or radiant condition of matter. We are quite familiar with the series of states of matter known as solid liquid and gaseous. To these Mr. Crookes adds a fourth term, the ultra-gaseous state, using it to designate an extremely rarefied form, such as for instance air would have if expanded to a volume twenty million times greater than its volume at the sea-level.

New and remarkable phenomena present themselves in a medium thus rarefied.

APPLICATION OF THE FOURTH DIMENSION TO GRAVITY, RADIANT ENERGY, AND THE CHEMICAL ELEMENTS.

We may now, following Mr. W. R. Ball, enquire very briefly whether the hypothesis of our existence in space of four dimensions throws any light upon certain difficulties or apparent inconsistencies in Physical Science. The cause of Weight or Gravity, the real nature of the Luminiferous star-revealing Ether, and the fact that there appear to be only a limited number of different kinds of matter (the Chemical Elements), all the atoms of each kind being (so far as we can tell) exactly alike, constitute difficulties, which our present knowledge is incompetent to solve.

Let us suppose that the various objects in our universe have, in addition to their usual three dimensions, a small and uniform thickness in the fourth dimension, a peculiarity which we can of course never actually discern. Suppose also that, in this fourth direction, our universe rests on a homogeneous elastic body, whose fourth dimensional thickness is small and constant. Then it seems possible to explain the transmission of radiant energy (heat and light), and of gravitational force, by the vibrations of this supporting body, without necessarily assuming the existence of any intervening medium.

It is suggested that the vibrations associated with radiant energy are transversal, those with gravity longitudinal. The velocity of transmission of gravitation has not yet been measured. If not infinite, it is at all events immensely greater than that of light, and hence a reason for Connecting gravitation with the longitudinal elasticity.

Next we may consider what explanation can be offered with regard to the limited number of elements, and the exact resemblance to each other of the atoms of any one element. A physical connection between all the molecules in the universe might account for these curious facts; and this connection would exist, if we assume a fourth dimension and the homogeneous elastic body before referred to; for upon this body all the molecules would rest. This body being elastic, it matters not in what fortuitous concourse the molecules upon it were set vibrating originally, eventually they would fall into certain groups, and all the molecules in each group would vibrate at the same rate. The above is a short resume' of Mr. Ball's highly ingenious speculation.

This hypothesis is suggested by a much simpler one, in which it is supposed that the Filmites of Flatland live upon the flat surface of an elastic solid, capable of transmitting vibrations, which vibrations constitute the light, electricity, and possibly the gravitation, of their world. The Filmites, being unable to perceive this solid, would probably invent an intervening surface medium or ether to account for the physical phenomena referred to. But the real link between the different portions of the Flatland universe would be the elastic solid, of which necessarily

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they were unconscious. If the supporting elastic solid be itself only a plane film, the diminution of intensity, as we recede from a source of disturbance, will be as the distance, and not as the square of the distance, from the source.

Mr. Hinton makes a very suggestive remark based upon analogy. Two bodies in contact have a common surface related to both. So our tridimensional bodies may be the contact of two quadridimensional bodies with each other, their common "surface" being a body of three dimensions. In a fluid the surface has remarkable properties due to what is called surface-tension, properties differing from those of the interior of the mass. Possibly the laws of our universe result from the surface tensions of the bodies of a universe of higher dimensions.

SPACE OF YET HIGHER DIMENSIONS.

Granting the possibility of a fourth dimension, we may assume as many more dimensions as we please. From the algebraical or analytical side there is a geometry, not merely of four, but of five, six, seven, up to N dimensions, where n is any number. Just as an equation with three variables may represent the boundary of a solid or tridimensional figure, and one with four variables that of the boundary of a body of four dimensions, so we may similarly find equations for spaces of yet higher dimensions.

Arthur Cayley, Sadlerian Professor in the University of Cambridge, is generally regarded as the greatest of living mathematicians. A portrait of him was painted by Dickenson, and the following lines addressed to the committee of the portrait fund, and written by the late Professor J. Clerk Maxwell, bear very appositely on the subject of this Paper.

"O wretched race of men to space confined!
"What honour can ye pay to him, whose mind
"To that which lies beyond hath penetrated?
"The symbols he hath formed shall sound his praise,
"And lead him on through unimagined ways
"To conquests new, in worlds not yet created.
"March on, symbolic host! with step sublime,
"Up to the flaming bounds of Space and Time!
"There pause, until by Dickenson depicted[1]
"In two dimensions, we the form may trace
"Of him whose soul, too large for vulgar space,
"In n dimensions flourished unrestricted."

According to Professor Cayley, the fundamental notion, pervading the whole of modern analysis arid geometry, is that of imaginary magnitude and imaginary space. The truths of mathematics are true precisely because they express the properties of purely imaginary objects, objects which after all, may be the only realities, the , compared with which the corresponding physical objects are as the shadows in the cave. If we had no conception of straightness, it would be meaningless to deny the existence, as of course we do, of a perfectly straight line in nature.

CONCLUSION.

I hope you will forgive me detaining you so long over a subject of so abstract, some would say of so useless, a character. But to me it seems one of the glories of the human intellect that it can rise into speculations so remote from things merely material. Placed between two Immensities, between two Eternities, we breathe for a brief period "an ampler ether a diviner air," and return refreshed to the common round the daily task. What a contrast between our tiny tridimensional bodies and our ever-expanding minds!

May not each of us say with Hamlet : -

"I could be bounded in a nutshell and count myself a king of infinite space."

Let me close by quoting a few lines from a poem by the great Poet, who so recently "has passed to where beyond these voices there is peace."

"What are men that He should heed us? cried the King of sacred song;
"Insects of an hour, that hourly work their brother insect wrong,
"While the silent Heavens roll, and Suns along their fiery way,
"All their planets whirling round them, flash a million miles a day.
"Many an Aeon moulded earth before her highest, man, was born,
"Many an Aeon too may pass when earth is manless and forlorn,
"Earth so huge, and yet so bounded-pools of salt, and plots of land-
"Shallow skin of green and azure-chains of mountain, grains of sand!
"Only That which made us, meant us to be mightier by-and-by,
"Set the sphere of all the boundless Heavens within the human eye.
"Sent the shadow of Himself, the boundless, through the human soul;
"Boundless inward, in the atom, boundless outward, in the Whole."

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