

UP, UP, AND AWAY!
The fifth dimension in physics

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The following essay contains abstract mathematical concepts that may tend to discourage the casual reader. However, anyone can read and comprehend this article even if they do not have the mathematical background necessary to understand the mathematical concepts that appear in the second section. There is enough information in the other sections that any interested reader can learn about Kaluza's theory by carefully reading the paper.

I: Historical Development

The first two attempts to derive a unified field theory as an extension to the General Theory of Relativity were made by Herman Weyl (1918)[1] and Theodor Kaluza (1921).[2] Over the years, each of these attempts has engendered one of the main groupings of unified field theories. Weyl sought to alter the geometry of the continuum[3] while maintaining the number of dimensions. His initial attempt was shown to yield physical consequences that were contrary to experimental evidence and thus was a failure. However, his work with affine connections was later extended by Sir Arthur Eddington, Albert Einstein and others, concluding with the work of Erwin Schrödinger who used the affine connection for his unified field theory in 1944. Einstein arrived at virtually the same theory as Schrödinger using his non-symmetric approach.[4] The Einstein-Schrödinger non-symmetric field,[5] which consists of the combined results of these two scientists' efforts toward unification, has come to represent what is considered to be the most advanced classical unified field theory.

On the other hand, Kaluza's theory kept the Riemannian space-time continuum of the General Theory of Relativity intact while extending the field structure by the addition of a fifth dimension. Like Weyl's theory, Kaluza's ideas have led to many extensions and modifications, but unlike Weyl's theory, the five-dimensional structure built by Kaluza has never been proven wrong and still stands as an independent theory as it was originally conceived. Although it has yet to be found completely unsound, without scientific merit, it has suffered from very serious criticisms that have hampered its credibility in the scientific community.

The original version of the theory appeared in 1921 in a paper entitled "Zum Unitätsproblem der Physik." The complete theory comprised only seven pages, but this was enough to capture the imagination of other scientists such as Einstein and Oskar Klein. This one article seems to be the major, as well as the only published contribution that Kaluza made to the five-dimensional theory. Any references to Kaluza's work always note only this paper. Yet there is evidence that Kaluza was working on this theory at least two years before his article was published. In a letter dated 21 April 1919, Einstein made the following remark regarding Kaluza's earlier communication of the five-dimensional idea to him: "... der Gedanke, dies (elektrischen Feldgrössen) durche eine fünfdimensionale Zylinderwelt zu erzeilen, ist mir nie gekommen und dürfte überhaupt neu sein. Ihr Gedanke gefällt mir zunnächst ausserordentlich."[6] This letter was written two years before Kaluza published his theory.

After his encouragement to Kaluza to pursue such an original idea, Einstein even went so far as to recommend Kaluza for a better academic position in 1926, recognizing the fact that Kaluza was far above the position of a 'privatdozent' that he held at the time in Königsberg. This overt support for Kaluza's work on a unified field theory may seem a contradiction considering that Einstein stated that he did not support other scientists' attempts to place electromagnetic theory and gravitation on a sounder footing than he had in his book *The Meaning of Relativity*, first published in 1922. "This inclusion of the theory of electricity in the scheme of the general theory of relativity has been considered arbitrary and unsatisfactory by many theoreticians. ... A theory in which the gravitational field and the electromagnetic field do not enter as logically distinct structures would be much preferable. H. Weyl, and recently Th. Kaluza, have put forward ingenious ideas along this direction; but concerning them, I am convinced that they do not bring us nearer to the true solution of the fundamental problem."[7]

Einstein himself made no active attempt to develop a unified field theory until 1923, at which time he attempted a theory based upon the affine connection which Weyl had originated and Eddington had modified.[8] Einstein's first written papers on the five-dimensional field theory did not appear until 1927[9] and these added nothing beyond Klein's development of the theory made the preceding year. So, despite his early encouragement and recognition of Kaluza's talent, it would have seemed that Einstein was never really convinced of the efficacy of the five-dimensional hypothesis. Einstein was strongly governed by the principle of parsimony in his assumptions, conceptual models and otherwise.

The five-dimensional idea hypothesis would appear to be solely the product of Kaluza, although there is one reference by Louis DeBroglie to the five-dimensional world suggested by Kaluza and Kramers.[10] And, except for an implication made by Wolfgang Pauli that Kaluza and Klein collaborated in some later modifications of the theory,[11] Kaluza seems to have worked alone and published nothing of any notoriety after his initial paper.[12] There were certainly earlier attempts to develop physical theories by utilizing either hyper-dimensional hypotheses or physical non-Euclidean geometries,[13] some of which predate general relativity by nearly a half century, but there is no evidence nor any suggestion that Kaluza had any prior knowledge of these theories.

In the literature regarding this subject, even the casual reader can discern an evolution of a 'main-line' of succession following Kaluza's original five-dimensional theory. Oskar Klein made the first important modifications of the five-dimensional theory and for many years attempted in different ways to use the fifth dimension as a convenient way to introduce the quantum of action into the gravitational field.[14] In 1930 Oswald Veblen and Banesh Hoffman sought to explain away the fifth dimension using projective geometry.[15] Their theory held that the fifth coordinate was only considered to be a projection of the four coordinates of real space. Important work was also done along this line of reasoning in the early 1930's by Pauli,[16] Van Dantzig and Schouten.[17] During this same period of time (1931) Einstein and Mayer[18] introduced a theory whereby each point of the Riemannian space corresponded to a four-dimensional flat space immersed in a flat five-dimensional space. In this case the four-dimensional Riemannian space was not embedded in a corresponding Riemannian five-dimensional space. This allowed Einstein and Mayer to introduce a five-dimensional tensor calculus with a corresponding five-dimensional space or coordinate system.

The next attempted modification to the theory came in 1938 by Einstein and Peter Bergmann.[19] In an attempt to give the fifth-dimensional component some real significance, the world was now considered closed with respect to the fifth dimension. This formed a kind of five-dimensional Riemannian structure. In 1941 Einstein and Bergmann, by then in collaboration with Bargmann,[20] showed that the integral-calculus equations representing the field in the 1938 theory could be replaced by a system of differential equations. This was Einstein's last published paper using the five-dimensional hypothesis. Several years later, Einstein left behind all attempts at establishing a unified field theory on the five-dimensional model and returned to his earlier non-symmetric derivation.

Another popular attempt at unification using the five-dimensional approach was made independently by Pasqual Jordan in 1945[21] and Y. Thiry in 1948.[22] These men allowed the fifth-dimensional component to vary instead of assuming it constant as had been done in all other five-dimensional theories. Jordan was able to relate his variable scalar to the gravitational constant introducing a new five-dimensional theory with fifteen field equations. J. Podolanski made one last and somewhat different modification in 1949.[23] In this theory a six-dimensional continuum was considered as a real physical space with a special laminated or sheet structure. By this point in time, all of the mainline[24] (popularly known) variations of the five-dimensional unified field theories seem to have been made. However, these variations of the five-dimensional hypothesis do not exhaust all of the work done along this line of reasoning. More recently, at least since about 1980, the Kaluza-Klein theory has been revived in attempts to derive supergravity theories and 'theories of everything,' as they are now called.[25]

All of these theories enjoy some notoriety and are often mentioned in the earlier popular literature dealing with five-dimensional theories. However, there are also a number of other theories and modifications based on the Kaluza model or the five-dimensional structure that are not as well known and have gone to a greater extent without the publicity enjoyed by the mainline theories. This is not to say that these other theories are without merit, nor is it to say that they may deserve more recognition than they have received. It is only to recognize them as legitimate attempts to describe the physical world.

As early as 1928, H.T. Flint[26] was trying to use the fifth dimension to account for quantum phenomena. These attempts were continued by Flint in dozens of published papers, at least until the mid-1950s. D. Meksyn[27] (1934) on the other hand extended the space-time continuum even further than Kaluza or Podolanski to include eight dimensions whose metric satisfied Einstein's law of gravitation, $G_{\mu\nu} = 0$. In 1935, G.Vranceanu[28] developed a theory based upon a non-holonomic hypersurface, which is locally four-dimensional with the local space-times at each point not tangent to the same four-dimensional space, thus utilizing a five-dimensional Riemannian space. Kentano Yano further developed the theory of Vranceanu.[29] And finally, H.C. Corben[30] has developed a simple unified field theory by extending the Maxwell-Lorentz equation of Special Relativity to five dimensions. These theories do not necessarily exhaust all of the attempts to derive a unified field based upon Kaluza's five-dimensional hypothesis, but they do serve to demonstrate that serious attempts to do so, which are not always recognized in the more easily obtained scientific literature, have been made by legitimate scientists. In fact, a great variety of attempts of this kind have been made either without success or with limited success.

II: Mathematical and Theoretical Structure

The Kaluza model was based solely upon Einstein's General Theory of Relativity in its original form and was unrelated to the quantum theory. The original purpose of Kaluza's work was merely to derive both Maxwell's electromagnetic formulas and Einstein's model of gravity from a single five-dimensional field. It was only after Klein modified Kaluza's theory a few years after its inception that quantum theory was included in the formulation of the five-dimensional model of space-time. So, in order to understand the problem as seen by Kaluza, a short discourse on General Relativity as it was formulated in the 1920's is in order.

The structure of the space-time continuum as described in the General Theory of Relativity can be summarized by the equation

$$R_{\mu\lambda} - \tfrac{1}{2} g_{\mu\lambda} R = -k T_{\mu\lambda} , \qquad \text{(I)}$$

where $\mu,\lambda = 1,2,3,4$. In laymen's terms, this formula simply means that the physical quantities representing matter and energy, found on the right side of the formula, can be represented by a curvature of the space-time continuum, which is represented by the quantities on the left side of the formula.

This equation does not explicitly include the electromagnetic field, only the gravitational field. In the classical sense, this is understood to mean that the presence of matter curves space-time, so the structure of space-time depends only on the presence of matter. However, many scientists assumed that electromagnetism must also play as important a role in the structuring of the space-time continuum as does gravity, if not a more important role than gravity. So, the derivation of the electromagnetic field itself seemed an important direction for research to follow.

Electromagnetism can be added to the defining equation of the space-time structure by simply adding a new term to Einstein's formulation.

$$R_{\mu\lambda} - \tfrac{1}{2} g_{\mu\lambda} R = -k(T_{\mu\lambda} + E_{\mu\lambda})\ .^{31} \qquad \text{(II)}$$

This addition of the electromagnetic term, $E_{\mu\lambda}$, seems to be rather artificial. It does not lead to simple solutions for charged particles in a combined electromagnetic and gravitational field in a way similar to that of an uncharged particle in the combined field. Instead of yielding a geodesic equation, as do particles which have mass and move in the gravitational field alone, a particle of mass m and charge e in such a combined field as that characterized by the gravitational potential $g_{\mu\nu}$, and the electromagnetic potential ϕ, would have a trajectory given by the equation:

$$\frac{d^2x^\beta}{ds^2} + \left\{{\beta \atop \nu\mu}\right\} \frac{dx^\lambda}{ds}\frac{dx^\mu}{ds} = \frac{e}{m} f^\beta_\alpha \frac{dx^\alpha}{ds}\ ,^{32} \qquad \text{(III)}$$

This result differs from the case of an uncharged particle only by the right hand side, which represents the covariant Lorentz force. In the case of an uncharged particle, the right side of the above formula becomes zero, or

$$\frac{e}{m} f^\beta_\alpha \frac{dx^\alpha}{ds} \rightarrow 0, \qquad \text{(IV)}$$

and the remaining equation is only the geodesic within the four-dimensional Riemannian metric.

The above equation (III) may not represent a geodesic in Riemannian space, but it can be regarded as a geodesic in a four-dimensional Finsler space. This Finsler space can be represented mathematically by its metric, such that

$$ds' = (g_{\mu\nu} dx^\mu dx^\nu)^{1/2} + \frac{e}{m} \phi_\mu dx^\mu\ .^{33} \qquad \text{(V)}$$

However, this representation also proves to be unsatisfactory for a combined field in space-time. Each new value of e/m yields a new Finsler space such that there is no single representation for the universe with all its different particles (and e/m ratios). So, there seems to be a problem with the artificial addition of a new component, representing electromagnetism, to the formula relating matter and the space-time curvature.

Within the Riemannian metric[34] there are only ten independent components, which can be used to represent the space-time structure. In the General Theory of Relativity the ten potentials needed to describe gravitation are identified with these ten independent components of the four-dimensional metric tensor $g_{\mu\nu}$,[35] leaving nothing to be identified with other potentials in the field such as electromagnetism. On the other hand the electromagnetic field can be described by four additional potentials, which are equivalent to a four-vector composed of the scalar potential ϕ and the three components of the vector magnetic potential A. So, fourteen potentials or independent components are needed to represent the combined field and this many components are unavailable within the original formulation of General Relativity. Since there are problems with adding a new electromagnetic component to the field equation, and the field equation itself does not have room to accommodate the extra variables needed for electromagnetism, then the only solution to this impasse seems to be to expand the field equation in some other manner.

In General Relativity, the field structure is determined by the Riemannian metric tensor, which is symmetric, and of rank 2. The number of field components for a tensor such as this is determined by the formula $N = ½(n)(n+1)$, where n is the number of dimensions. Riemannian space, being four-dimensional, then has N=10 components. However, the Riemannian structure can still be saved and an adequate number of independent components found by increasing the number of dimensions to five. This gives fifteen components, one more than is needed for the combined field. Kaluza was the first to see this as a possible answer to the problem of developing a unified field structure incorporating electromagnetism and gravitation on an equal basis within the geometric structure of the space-time continuum.

By introducing the fifth component in space-time, Kaluza was able to give a geometric representation to the generally covariant form of Maxwell's electrodynamics.[36] Whereas there had not been enough components in the four-dimensional space-time to incorporate electromagnetism, when the five-dimensional space is used there are one too many components of the metric tensor. This represented Kaluza's first problem in using the five-dimensional Riemannian tensor. Another important problem in a theory of this type is that such a theory must account for the fact that all empirical evidence is four-dimensional in nature rather than five-dimensional.

Other scientists have dealt with these two problems in various ways, but Kaluza simply added no direct physical significance to his five-dimensional hypothesis,[37] merely using it as a mathematical tool. He further assumed that the field variable $g_{\mu\nu}$ would not depend on the fifth coordinate, but only on the four coordinates of an ordinary space-time continuum when a suitable coordinate system was chosen.[38] This last assumption allowed the fifteenth component to be ignored while allowing for a somewhat "atrophied" fifth dimension.[39] The five-dimensional hypothesis of Kaluza could then be seen to be somewhat limited in that he was not dealing with as general a five-dimensional structure as was possible. Other scientists have modified Kaluza's theory (Einstein and Mayer, 1931; Einstein and Bergmann, 1938; Einstein Bergmann and Bargmann, 1941) to make it more general by altering Kaluza's second assumption.

To begin this theory it is necessary to consider a five-dimensional space, which is defined by the metric, or line element,

$$d\sigma^2 = \gamma_{\mu\nu} dx^{\mu} dx^{\nu} \,, \tag{VI}$$

where μ,ν now becomes equal to 0,1,2,3, or 4 to encompass the fifth component of the field. Using either a 0 of a 5 can represent the fifth coordinate, depending on the author of any given article. This metric must be made cylindrical by way of the fact that the field variables are independent of the fifth dimension, or rather more simply, we are dealing with spaces having some sort of axis of symmetry around which the fifth coordinate is measured.[40] In the particular coordinate system used in this theory the condition of cylindricity can be described by

$$\frac{\delta \gamma_{\mu\nu}}{\delta x_5} = 0 \,.^{41} \tag{VII}$$

As stated above, this means that there is no variation in $\gamma_{\mu\nu}$. It is constant with respect to the fifth coordinate.

Analytically, this cylindrical condition can be put in a slightly different form. If an arbitrary curve in the five-dimensional space, with endpoints P_1 and P_2, is infinitesimally displaced along the A-curves,[42] the length of the arbitrary curve will not change. The infinitesimal displacement is defined by a small amount $\delta^{\nu}{}_{\chi}$, whereby $\delta^{\nu}{}_{\chi} = \tau \cdot A^{\nu}$.[43] The quantity τ is an infinitesimal constant displacement and A^{ν} is the contravariant vector representing the five-dimensional curve. The length of the metric (as defined above) can only remain constant when the vector A^{ν} satisfies Killing's equation,

$$A_{\mu;\nu} + A_{\nu;\mu} = 0 \,.^{44} \tag{VIII}$$

In this case, the subscript μ refers to the vectors A^{μ} or the "A-lines" which are tangent to the vectors A^{ν}, which form a vector field A^{ν}. The fact that Killing's equation is satisfied is enough to show that the five-dimensional components are constant and thus the cylindrical condition satisfied. Some authors refer to this condition as A-cylindricity.

As another consequence of Kaluza's theory, "the lines to which the A^{μ} are tangents - the 'A-lines' - have to be geodesics."[45] Therefore, the A-lines must conform to the geodesic equation

$$\frac{d^2x^{\nu}}{d\sigma^2}+\Gamma^{\nu}_{\alpha\beta}\frac{dx^{\alpha}}{d\sigma}\frac{dx^{\beta}}{d\sigma} = 0 . \quad \text{[46]} \tag{IX}$$

The A-lines themselves are defined by

$$\frac{dx^{\nu}}{d\sigma} = \lambda A^{\nu}, \tag{X}$$

where λ is defined by $1/\lambda^2$ being equal to the norm of A. The cylindricity condition shows that the A-lines remain constant within the vector field A^{ν}, but implies nothing about the A-lines for the whole of space. The requirement that the A-lines are geodesics immediately leads to a complete space structure whereby the norm of A is constant throughout all of space, not only along the A-lines. Both the cylindricity condition and the subsequent constancy of the A-lines are important factors in the five-dimensional unified field theory. These conditions lead to the proper transformations within a special coordinate system, whereby both the equations of gravitation, as found in the General Theory of Relativity, and the Maxwell-Lorentz equations can be derived.

Kaluza allowed his five-dimensional space-time structure to be based upon a special coordinate system. This structure consisted of a four-dimensional width of quantum space-time that cut each of the A-lines only once (the A-lines represent the five-dimensional component). The distance along any A-line starting from the one endpoint (which equals 0), according to the original equation for the metric, would be

$$x^0 = \int_0^{x^0}\sqrt{\gamma_{00}}\,dx^0, \tag{XI}$$

therefore giving a value for γ_{00} of unity.[47] The condition that $\gamma_{00} = 1$ can also be regarded as a normalization of the quantum. This also leads to a value of $A^0 = 1$.

Choosing a value of (+)1 for γ_{00} has two effects. First of all it "normalizes" the other components,[48] $A^1...A^4$ representing the other four dimensions, such that they vanish and it is found that

$$\gamma_{\mu\nu\rho} = 0. \tag{XII}$$

This statement means that all of the factors dependent on the x^0 component disappear, or rather; any components or field variables having a combination of indices with 0 and either 1,2,3 or 4 will disappear. This disappearance guarantees that there will be no physical evidence of the fifth dimension in four space-time, while reducing the total number of variables by one, from fifteen to fourteen, just the number necessary to represent both gravitation and electromagnetism. Also, the choice of a "positive" number to represent the γ_{00} allows for a "space-like" character of the fifth dimension.[49] This planned limitation to Kaluza's five-dimensional space-time has led to criticisms that the choice of $\gamma_{00} = 1$ was "rather arbitrary (though convenient)"[50] and that the space-like character of the fifth dimension is not a certainty. Some later theorists dropped this limitation to allow for a new theory of the space structure, such as Jordan and Thiry's theory, which utilized all fifteen variables.

Once this special coordinate system is established, it then becomes necessary to ask what kind of transformations can be made while preserving the special character of the system and without changing the A-lines. It is necessary that the coordinate system used be invariant to transformations. In this way the character of the space structure can be shown to be the cause of the apparent physical phenomena, as in the General Theory where gravitational phenomena became geometrical properties of the space-time structure under invariant transformations. For the special geometry chosen by Kaluza to represent his five-dimensional space-time, two different transformations can be made which leave the A-lines invariant. The first set of transformation laws, called the "four-transformation," is

$$\overline{x^0} = x^0, \quad \overline{x^a} = f^a(x^1...x^4), \tag{XIII}$$

where a = 1,2,3,4 and the line represents the transformed coordinate. The quantity f^a is a function of the x^a. The second set of transformations, called the "cut-transformation," is

$$\overline{x^a} = x^a, \quad \overline{x^0} = x^0 + f^a(x^1...x^4). \tag{XIV}$$

The four-transformations represent the general coordinate transformations that are common to General relativity, but the cut-transformations are specific to the special coordinate system used in Kaluza's theory.[51]

The metrical coefficients transform according to the formulation

$$\gamma_{\mu\beta}' = \gamma_{\mu\nu}\frac{\delta x^{\mu}}{\delta x^{\alpha'}}\frac{\delta x^{\nu}}{\delta x^{\beta'}} \quad ,^{52} \qquad \text{(XV)}$$

which gives the following values for both γ_{00} and γ_{0i} under transformation:

$$\gamma'_{00} = \gamma_{\mu\nu}\frac{\delta x^{\mu}}{\delta x^{01}}\frac{\delta x^{\nu}}{\delta x^{01}} = \gamma_{00}. \qquad \text{(XVI)}$$

In other words, γ_{00} is invariant, non-changing, under the indicated mathematical transformation, as it should be. The components γ_{0i} are similarly invariant. Furthermore, the field components A_m vary only by an additive constant under the same transformations, such that

$$\overline{A_m} = \gamma'_{0m} = \gamma_{\mu\nu}\frac{\delta x^{\mu}}{\delta x^{0'}}\frac{\delta x^{\nu}}{\delta x^{m'}}$$

$$= \gamma_{0m}\frac{\delta x^{i}}{\delta x^{m'}} + \gamma_{00}\frac{\delta \psi^{0}}{\delta x^{m'}}$$

$$= \gamma_{0m} + \frac{\delta \psi_{0}}{\delta x^{m'}} \quad . \qquad \text{(XVII)}$$

From this, it can be clearly seen that the A_m behave in a way similar to the electromagnetic four-vector in that when they are transformed there is a gradient added to the transformed vector. "This corresponds to the fact that the electromagnetic potentials are defined only up to additive terms which are gradients of an arbitrary function."[53]

In summing up the effects of these transformations, we deal with three types of field variables; the γ_{mn}, A_m (= γ_{0m}) and γ_{00}. The γ_{mn} correspond to the sixteen components of a metric representing the four-dimensional space-time continuum. In general relativity these reduce to the ten components that describe gravitation. In the five-dimensional theory the variables γ_{mn} act as a four-tensor under a four-transformation and are invariant under the cut-transformations. Thus they are equivalent in the five-dimensional representation to the general relativistic representation. The A_m (= γ_{0m}) correspond to the eight (four + four) terms in a metric (five by five), which represent the mixed terms of the fourth and fifth dimensions. Upon a four-transformation, the A_m act as a four-vector while the cut-transformation allows them to vary only by an additive term, such that

$$\overline{A_m} = A_m - \frac{\delta f}{\delta x^m}\ .$$[54]

(XVIII)

This formula is equivalent to the above equation XVII, which allows for the introduction of the electromagnetic four-vector into the space-structure if the A_m are equated to the electromagnetic potentials ϕ_i by analogy.

The last term γ_{00} is a purely fifth-dimensional component and under both transformations is seen to be a scalar and invariant. This allows the field variable γ_{00} to be set equal to +1, as Kaluza has done, and thus it is no longer variable and does not affect the field structure. This mathematical model demonstrates that the space-structure, as it is outlined here, is a product of both the electromagnetic and gravitational fields. "The five-dimensional space structure is equivalent to a four-dimensional one with a metric (g_{mn}) and a vector field (A_m) determined only up to an additive arbitrary gradient."[55]

It is possible to define a new field quantity $g_{\mu\nu}$ whose components are those of a five-dimensional tensor. Taking the results of the above transformation and applying them to the definition of the metric given earlier, we get[56]

$$d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu$$

$$= \gamma_{00} dx^0 dx^0 + \gamma_{0i} dx^0 dx^i + \gamma_{i0} dx^i dx^0 + \gamma_{ik} dx^i dx^k$$

$$= (dx^0)^2 + 2\gamma_{0i} dx^0 dx^i + \gamma_{0i}\gamma_{0k} dx^i dx^k - \gamma_{0i}\gamma_{0k} dx^i dx^k + \gamma_{ik} dx^i dx^k$$

$$= (dx^0 + \gamma_{0i} dx^i)^2 + (\gamma_{ik} - \gamma_{0i}\gamma_{0k}) dx^i dx^k$$

$$= (dx^0 + \phi_i dx^i)^2 + (\gamma_{ik} - \phi_i \phi_k) dx^i dx^k$$

$$= (dx^0 + \phi_i dx^i)^2 + g_{ik} dx^i dx^k \quad ,^{57} \qquad \text{(XIX)}$$

where we have let $\gamma_{0i} \Longrightarrow \phi_i$ and $(\gamma_{ik} - \phi_i \phi_k) \Longrightarrow g_{ik}$. In general, when speaking of a five-dimensional space-time, the subscripts i,k refer to the normal four-dimensional space-time while the subscripts μ,ν refer to the five-dimensional counterpart of our normal space-time. So these variables g_{ik} correspond to the usual metrical coefficients in general relativity whereby the Riemannian metric is given by $ds^2 = g_{ik}\, dx^i dx^k$. The other term represents an electromagnetic contribution to the metric. Since A_m ($= \gamma_{0i}$) behaved like electromagnetic potentials under a cut-transformation it is easy to assume them equal to the electromagnetic potential ϕ_i, apart from a constant factor of proportionality.[58] Since the additive scalar evolved during transformation is sometimes referred to as a "gauge," that transformation is sometimes called a "gauge-transformation," but this should not be confused with a Weyl type of "gauge-transformation."[59] There is also an anti-symmetric tensor involved with this transformation, which is left invariant. This tensor can be evaluated as

$$A_{m,n} - A_{n,m} = \frac{\delta \gamma_{k0}}{\delta x^i} - \frac{\delta x_{i0}}{\delta x^k} = f_{ik}$$

with i,k = 1,2,3,4. It is proportional to the electromagnetic field strength. The new line element is thus shown to be dependent on both electromagnetic and gravitational contributions and the two groups of terms, which add to give the line element $d\sigma^2$, become infinitesimal invariants in the field.

One of the problems, which were evident in classical General Relativity using a Riemannian four-dimensional space, was that particles did not travel along an ordinary geodesic in the combined electromagnetic-gravitational field. If this new theory of Kaluza's is valid, then it should correctly show that the trajectory of charged particles is a geodesic within the new model of the combined field. Keeping this in mind, once the line element $d\sigma^2$ has been found, the geodesics of a particle's trajectory can be found. If τ is introduced as an arbitrary parameter, such that $d\sigma/d\tau = L$, then

$$L^2 = g_{ik}\frac{dx^i}{d\tau}\frac{dx^k}{d\tau}+(\frac{dx^0}{d\tau}+\phi_i\frac{dx^i}{d\tau})^2$$

$$= \gamma_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \qquad \text{(XXI)}$$

The geodesic must be the shortest path in this space and therefore the geodesic equations must conform to a variational principle such that

$$\delta\int L d\tau = 0 \quad .^{60} \qquad \text{(XXII)}$$

The geodesic equations that conform to this condition are of the form

$$\frac{\delta L}{\delta x^\nu}-\frac{d}{d\tau}(\frac{\delta L}{\delta \dot{x}^\nu}) = 0 \ , \qquad \text{(XXIII)}$$

where x denotes a speed, or rather the 'dot' denotes a differentiation with respect to time. Two cases of solutions arise here. The first, where $\nu = 0$, represents the fifth coordinate and the second, where $\nu = \phi$, represents the other four coordinates. In the first case we have,

$$\frac{d}{d\tau}[\frac{1}{2L}\cdot 2(\dot{x}^0+\phi_i\dot{x}^i)] = 0. \qquad \text{(XXIV)}$$

Setting $\tau = s$ so that L becomes constant, we arrive at a solution of

$$\dot{x}^0+\phi^i\dot{x}^i = constant = B. \qquad \text{(XXV)}$$

Solving further yields

$$L^2 - B^2 = constant = g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} \quad . \tag{XXVI}$$

The constant here can be normalized to -1.[61] Another term also turns out to be constant for the geodesics, such that

$$\frac{dx^0}{ds} + \gamma_{i0}\frac{dx^i}{ds} = constant. \tag{XXVIII}$$

These terms are independently constant for a suitable choice of the parameter s. For the second case where $\gamma \neq 0$, the geodesic equation yields,

$$g_{jk}\ddot{x}^j + \left[{ij \atop k}\right]\dot{x}^i\dot{x}^j + B \cdot f_{ik}\dot{x}^j = 0. \tag{XXIX}$$

Here the term f_{kl} is the electromagnetic field strength

$$f_{jk} = \left(\frac{\delta\phi_k}{\delta x^j} - \frac{\delta\phi_j}{\delta x^k}\right).$$

This equation then becomes, by a suitable multiplication while expressing the δx^j as dx^j/ds,

$$\frac{d^2x^l}{ds^2} + \left\{{ij \atop l}\right\}\frac{dx^i}{ds}\frac{dx^j}{ds} + Bf_j^l\frac{dx^j}{ds} = 0. \tag{XXX}$$

This particular equation is very nearly the equation of motion stated earlier for the trajectory of a charged particle in a combined field. It can be assumed that the constant B is equivalent to the ratio e/m_o, which can be justified since B is invariant and constant under transformation. This equation correctly describes the trajectory of a charged particle in the combined field. So it would seem that Kaluza's theory yields electromagnetic results that cannot be derived from General Relativity alone. It was earlier stated that resorting to a Finsler space in the case of General Relativity could only form the above equation, but that this was physically impossible to do on a consistent basis. However, the five-dimensional theory easily accounts for this equation, further reducing electromagnetism to a property of the five-dimensional space structure.

The field equations in a vacuum can be found using a simple variational principle. Kaluza assumed that the Lagrangian to be used in this variational principle was the product of the five-dimensional curvature scalar R and the square root of the determinant $|\gamma_{\alpha\beta}|$, or

$$\delta \int R\sqrt{(-\gamma)}\,d\vec{x} = 0, \qquad \text{(XXXI)}$$

where the dx can be taken either over four dimensions, or, by the cylindricity condition, over five dimensions. The curvature scalar R is found to be,[62]

$$R = \delta^{i}_{h} g^{kl} R_{ikl} + \tfrac{1}{4} A_{\rho\sigma} A^{\rho\sigma} \qquad \text{(XXXII)}$$

$$= \delta^{i}_{h} g^{kl} R^{n}_{ijk} + \tfrac{1}{4} \phi_{\rho\sigma} \phi^{\rho\sigma}.$$

This curvature scalar has been somewhat limited by the restriction that the $A_{\rho\sigma}$ (= A_m) variables can only enter the term through their skew symmetric derivatives. These are the only parts that do not vanish during the act of derivation. Kaluza's variation is therefore expressed as

$$\delta \int (\delta^{i}_{h} g_{kl} R^{n}_{ikl} + \tfrac{1}{2} \phi_{rs} \phi^{rs}) \sqrt{-(g)}\,d\vec{x} = 0 \quad .^{63} \qquad \text{(XXXIII)}$$

It is restricted to the conditions that $(\delta g^{rs})_{,5} = 0$ and $(\delta g_r)_{,5} = 0$, or rather $\gamma_{ik,0} = 0$ and $\gamma_{00} = \alpha$. The field equations then found are

$$R^{\mu\lambda} - \tfrac{1}{2} g^{\mu\lambda} R + \tfrac{1}{2} \alpha\beta^{2} E^{\mu\lambda} = 0 \quad ,^{64} \qquad \text{(XXXIV)}$$

and

$$\frac{\delta[\sqrt{(-g)}\, f^{\mu\lambda}]}{\delta x^{\mu}} = 0. \qquad \text{(XXXV)}$$

These field equations are the same as those of the General Theory of Relativity with an electromagnetic field and the Maxwell-Lorentz equations, if we merely set $k = \frac{1}{2}\alpha\beta^2$.[65]

In this manner, the Maxwell-Einstein theory can be formally derived from the unified five-dimensional geometrical framework developed by Kaluza without any modifications to the already established theories of electromagnetism and gravitation. In the original form of the Kaluza theory, the fourteen field equations were found to match the Maxwell-Einstein equations only in the first approximation. However, "It was soon shown that this equivalence was not only approximate, but rigorous, if the proper combinations of metric components were accepted as the gravitational potentials."[66] The theory, as stated above, is actually a more advanced version developed by Klein in 1926. For this reason, some authors refer to it as the Kaluza-Klein theory rather than giving the credit to Kaluza alone.

Part II

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